Quiz 4 - Sections 6 and 9

Fall 2012

1. (6 points) Find all the local maxima, local minima, and saddle points of the function $f(x, y) = 4 + x^3 + y^3 - 3xy$.

Solution: We first examine the gradient:

$$\nabla f(x, y, z) = \langle 3x^2 - 3y, 3y^2 - 3x \rangle.$$

This vector is zero when both components are zero, i.e.:

$$\begin{aligned} x^2 - y &= 0 \\ -x + y^2 &= 0 \end{aligned}$$

Using $y = x^2$ in the second equation we find

$$-x + x^4 = 0$$

which only has two solutions (in the reals): $x = 0$ and $x = 1$. Therefore, the only critical points are

$$(0, 0) \quad \text{and} \quad (1, 1).$$

We now examine the Hessian:

$$\mathcal{H}f = \begin{pmatrix} 6x & -3 \\ -3 & 6y \end{pmatrix}.$$

When $x = y = 0$ this has determinant equal to -9, so the point $(0, 0)$ is a saddle point of f. When $x = y = 1$ the determinant is equal to $36 - 9 = 25 > 0$ and $6x = 6 > 1$, and so the point $(1, 1)$, where $f(1, 1) = 3$, is a local minimum of f.

2. (4 points) Show how to reverse the order of integration in $\int_0^2 \int_x^2 f(x,y)\,dy\,dx$.

Solution: We are integrating over the triangle with endpoints $(0,0)$, $(0,2)$ and $(2,2)$, thus (by Fubini' theorem) we can integrate first with respect to x from 0 to y and then with respect to y from 0 to 2:

$$\int_0^2 \int_x^2 f(x,y)\,dy\,dx = \int_0^2 \int_0^y f(x,y)\,dx\,dy.$$