Some common common mistakes

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1 Domains and Range

To find the domain of a function you have to look for "dangerous" operations (this are operations for which the domain is not every real number.) At this level the most basic operations which could be potentially dangerous are

1. **Division**: If you divide make sure that the denominator is not 0.
2. **Square root**: The square root of a negative number is not defined (yet) so make sure that what ever is inside your square root is *greater than or equal* to 0.
3. **Logarithms**: So far we only know how to define the logarithm (of any base) of a number if it is *strictly greater* than 0.

If your function contains any of this operations in its definition you have to make sure you don't do any of what's above. The set of points where every part of your function is defined is called the domain.

Example. Let's say we want to find the domain of $f(x) = \frac{\sqrt{x+y}}{x-y}$. We immediately see that the dangerous operations are the square root and division. We start with the square root: the domain cannot contain the set of points (x, y) where $x + y < 0$, so the domain will be some subset of $x + y \geq 0$.

Now we take a look at the division, we are dividing by $x - y$ so the domain cannot contain the set of points where $x - y = 0$, that is, the set of points where $x = y$. We know from before that *on top of this* $x + y \geq 0$, so the domain has to be the set of all points (x, y) such that

$$x \neq y \quad \text{AND} \quad x + y \geq 0.$$

Example. Suppose now that we are given the function $f(x, y) = \frac{1}{\log(x+y)}$. Here we are dividing and taking logarithms. Let's start with the logarithm: we have to make sure that $x + y > 0$, so the domain will be some subset of this.

Now take a look at the division: we are dividing by $\log(x+y)$ so we make sure that $\log(x+y) \neq 0$. The function $\log(z)$ is 0 precisely when $z = 1$, so in this case $x + y$ cannot be 1 for (x, y) to be in the domain.

So we have two necessary conditions for the point (x, y) to be in the domain of f:

$$x + y \neq 1 \quad \text{AND} \quad x + y > 0.$$

Since there aren't any other dangerous operations these conditions are necessary and sufficient. We can actually describe this domain geometrically: it is just the part of $\mathbb{R}^2$ with $y \geq -x$ and without the line $y = 1 - x$.

Now how do we find the range of a function? There is no procedure that will work every time but this is generally what you do:

First, what is the range? It is the set of *values* that a function can give you back. So if you find an x for which $f(x) = 234$ then 234 is in the range of f.

Example. Find the range of the function $f(x) = \log x$.

First we recall that the domain is the set $x > 0$. We can use the definition of logarithm to see that $\log(e^x) = x$, so the point x is in the **range** of f whenever e^x is in the domain of f, that is, always since $e^x > 0$. We can do this for every x in $\mathbb{R}$ so, for the same reason, every $x \in \mathbb{R}$ is in the range of f. In this case we can finish here since the range will not be able to contain anything else and thus the range is $(-\infty, +\infty)$.

Suppose now that you found that $f(1) = 234$ and that $f(2) = 543$ and that f is **continous** in $[1, 2]$, then f must take every value between 234 and 543 (this is known as the Intermediate Value Theorem.)

This idea leads us to a procedure that works *most* of the time: if a function is continuous on a connected set (we haven't defined precisely what connected means, but right now you can just assume it is what you think it is) and m, M are, respectively, the minimum and maximum values that f takes in this interval, then the range of f contains the interval $[m, M]$. We will apply this idea to some examples:

Example 1.1. Find the range of the function $f(x) = 4 - x^2$ for $x \in [-3, 7]$.

What is the maximum value of f in $[-3, 7]$? We can do two things: one way to do this is the usual: take derivatives, find critical values, etc. But in this case it is easier since we know that f is a downwards parabola centered at 0, so its global maximum value is $f(0) = 4$. The point 0 is in $[-3, 7]$ so the maximum of f in $[-3, 7]$ is 4.

What about the minimum? Again, since f is a downwards parabola, we just have to check what is the minimum at the boundary of $[-3, 7]$: $f(-3) = 4 - 9 = -5$ and $f(7) = 4 - 49 = -45$. Thus the minimum value is -45 and hence the range is $[-45, 4]$.

Let's do one last example:

Example 1.2. Find the range of the function $f(x, y) = \log(9 - x^2 - y^2)$.

We have to be careful here because the domain is non-trivial: $9 - x^2 - y^2$ has to be > 0 so $x^2 + y^2 < 9$, which is an open disk of radius 3 centered at the origin. What is the largest value f can take? We first recall that the logarithm can take every real number so we want to use this at our advantage. In particular $\log(z)$ tends to $-\infty$ when $z \to 0$ and $\log(z) \to +\infty$ when $z \to \infty$, so the minimum value of $\log(9 - x^2 - y^2)$ will be attained when $9 - x^2 - y^2$ is smallest. In this case $9 - x^2 - y^2$ ranges from 9 all the way down to 0 but not including 0 because this will throw us outside the domain. In this case there is no minimum value because f be an arbitrarily big negative number but it never is $-\infty$ (remember that $\log(0)$ is not defined.) But this is good enough for us.

On the other hand the maximum value of f is attained when $9 - x^2 - y^2$ is largest, that is when it is 9. This attained when $(x, y) = (0, 0)$ so the largest value of f is $f(0, 0) = \log(9)$.

Thus f ranges from $-\infty$ all the way up to $\log(9)$, so the range is $(-\infty, 9]$.

Be careful not to confuse the domain and the range.

2 Continuity and Limits

Suppose you are given a function f of two variables and you have to find the limit (if it exists) as $(x, y) \to (x_0, y_0)$. You first have to check that the limit exists, to do this you first try to see if it *doesn't* exist. The limit won't exist if we can find two paths to (x_0, y_0) with different limits. If you find two such paths and the limits do not agree then the limit does not exist and your job is done.

Suppose that you tried two paths and the limits were the same. You may be tempted to conclude that then the limit exists and equals any of these (equal) limits, but this is **WRONG**, there may be another path through which the limit is different. There may even be infinitely many paths with the same limit and we still wouldn't be able to conclude that the limit exits.

In most cases, if you approach the limit point through two or three different paths and the limits are equal then there is a high chance (when doing exercises) that the limit exists. Suppose that $(x_0, y_0) = (0, 0)$ and that you approached the limit along the X axis, along the Y axis, and along the diagonal and all three limits coincide. This may have given you hopes that the limit exists so now you have to prove this. So what do you do this? Sadly there is no answer that works every time, you just *get used to it.*

When doing exercises the most common case where a limit may non-trivially exists is the famous $\frac{0}{0}$ case. The idea here is that you can probably factor out the "zero part" of the denominator, the one that is making it 0, and with some luck something in the numerator will cancel it out.

Example 2.1. Find $\lim_{(x,y)\to(0,0)} \frac{x^2y+y^2x}{xy+x^2y}$.

Following the idea of factoring everything as much as possible we end up with

$$\frac{x^2y + y^2x}{xy + x^2y} = \frac{xy(x + y)}{xy(1 + x)},$$

so

$$\lim_{(x,y)\to(0,0)} \frac{x^2y + y^2x}{xy + x^2y} = \lim_{(x,y)\to(0,0)} \frac{xy(x + y)}{xy(1 + x)} = \lim_{(x,y)\to(0,0)} \frac{x + y}{1 + x} = 0.$$

Other useful techniques are multiplying and dividing by the conjugate, which may simplify things, and using the Sandwich Theorem.

Example 2.2. Find $\lim_{(x,y)\to(0,0)} xy\frac{x^2-y^2}{x^2+y^2}$

One first look at this tells us that we are dealing with the infamous $\frac{0}{0}$ limit. We tried several paths and all were 0 so we are led to believe that the limit exists. That the limit is 0 means that the numerator is smaller than the denominator in the limit point, of course both are 0 but one is a "smaller" 0 than the other. To exploit this idea we try to factor something which tends to 0 and hope that what's left is bounded, then an application of the Sandwich Theorem will yield that the limit is indeed 0.

The xy is already factored for us so we could try to see if $\frac{x^2-y^2}{x^2+y^2}$ is bounded by some constant M near 0. If we try $M = 1$, we have to check that $|x^2 - y^2| \leq x^2 + y^2$ and this is true (it is just the triangle inequality.)

Thus we have

$$-|xy| \leq xy\frac{x^2 - y^2}{x^2 + y^2} \leq |xy|,$$

so by the Sandwich Theorem we have

$$\lim_{(x,y)\to(0,0)} xy\frac{x^2 - y^2}{x^2 + y^2} = 0.$$