LORENTZ SPACES AND REAL INTERPOLATION
THE KEEL-TAO APPROACH

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1. Introduction

If an operator T is bounded on two Lebesgue spaces, the theory of complex interpolation allows us to deduce the boundedness of T "between" the two spaces. Real interpolation (of L^p spaces) provides, in a way, a refinement of complex interpolation, this will be shown as we progress. We will follow Tao's notes [6]: in particular, we claim no originality over any argument here.

Consider the (centered) Hardy-Littlewood maximal operator:

$$Mf(x) = \sup_{r>0} \int_{B_r(x)} |f(y)|\, dy, \tag{1}$$

defined for $f \in L^1_{\text{loc}}$. We would like to obtain L^p boundedness of this operator for $1 \le p \le \infty$, but alas, this is not the case: although it is trivially bounded from L^∞ onto itself, it is **not** bounded on L^1:

Proposition 1.1. *If $f \in L^1(\mathbb{R}^d)$ and f is not identically 0 then*

$$Mf(x) \ge \frac{C_d}{|x|^d}$$

for some constant C_d dependent on f and the dimension, in particular $f \notin L^1(\mathbb{R}^d)$.

Proof. Since f is not identically 0, there exists an $R > 0$ such that

$$\int_{B_R(0)} |f| \ge \delta > 0.$$

Now, if $|x| > R$, $B_R(0) \subseteq B(x, 2|x|)$, so

$$Mf(x) \ge \frac{1}{|B(x, 2|x|)|} \int_{B_R(0)} |f| \ge C_d \frac{\delta}{|x|^d}.$$

□

Thus M is indeed unbounded as an operator from L^1 onto itself. However, We do have the following weaker form of L^1 boundedness; proved by Hardy-Littlewood in [1] for $d = 1$ and by Wiener in [7] for $d > 1$:

Theorem 1.2. *There exists a constant $C > 0$ such that for all $f \in L^1$*

$$m(\{x \in \mathbb{R}^d \,:\, Mf(x) > \lambda\}) \le C \frac{\|f\|_{L^1}}{\lambda},$$

where m is the Lebesgue measure on $\mathbb{R}^d$.

Observe that, by Chebychev's inequality, we would have had this bound if M had been bounded on L^1, so it is indeed a weaker statement.

This leads us to study the space of functions satisfying Chebychev's inequality:

$$\mu(\{x \in X \,:\, |f(x)| > \lambda\}) \leq \frac{\|f\|^p_{L^p(X,\mu)}}{\lambda^p}. \tag{2}$$

More precisely, define the *weak-L^p* space $L^{p,\infty}(X,\mu)$ to be the set of all measurable functions f for which its $L^{p,\infty}$ quasi-norm

$$\|f\|_{L^{p,\infty}} = \sup_{\lambda>0} \lambda\mu(\{x \,:\, |f(x)| > \lambda\})^{1/p} \tag{3}$$

is finite. To abbreviate we will use the notation

$$d_f(\lambda) = \mu(\{x \in X \,:\, |f(x)| > \lambda\})$$

when there is no confusion as to what measure we are using. Also, we define by convention $L^{\infty,\infty}$ to be L^∞.

The following lemma is a simple application of the monotone convergence lemma:

Lemma 1.3. *Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of measurable functions such that $|f_n(x)| \leq |f_{n+1}(x)|$ μ-almost everywhere and for all $n \in \mathbb{N}$. If we have $|f_n| \to f$ μ-almost everywhere for some function f then*

$$d_{f_n}(\lambda) \nearrow d_f(\lambda)$$

when $n \to \infty$ for all $\lambda > 0$.

As we have seen, Chebychev's inequality gives us the inclusion $L^p \hookrightarrow L^{p,\infty}$ for $p > 0$. This inclusion is proper as the following example shows:

Example 1.4. If $f(x) = |x|^{-\frac{d}{p}}$ then

$$\int_{\mathbb{R}^d} |f(x)|^p \, dx = \int_{\mathbb{R}^d} |x|^{-d} \, dx = \infty,$$

however

$$\sup_{\lambda>0} \lambda d_f(\lambda)^{1/p} = \nu_d^{1/p} < \infty,$$

where ν_d is the measure of the unit ball in $\mathbb{R}^d$.

As we have noted, $\|\cdot\|_{L^{p,\infty}}$ is not a priori a norm, but only a quasi-norm:

Proposition 1.5. *We can easily see that*

$$d_{f+g}(\lambda) \leq d_f(\lambda/2) + d_g(\lambda/2),$$

which is just the fact that when two terms sum more than λ, at least one of them has to be greater than $\lambda/2$. Using this fact we have:

$$\begin{aligned} \|f+g\|_{L^{p,\infty}} &= \sup_{\lambda>0} \lambda d_{f+g}(\lambda)^{1/p} \\ &\leq \sup_{\lambda>0} \lambda \left(d_f(\lambda/2) + d_g(\lambda/2)\right)^{1/p} \\ &\leq 2C_p \left(\|f\|_{L^{p,\infty}} + \|g\|_{L^{p,\infty}}\right). \end{aligned}$$

We will generalize the weak-L^p spaces a bit further, bit first we need the following well-known identity:

Proposition 1.6. *Let $0 < p < \infty$ and f be a measurable function, then*

$$\|f\|_{L^p} = p^{1/p} \left(\int_{\mathbb{R}^+} \lambda^p d_f(\lambda) \, \frac{d\lambda}{\lambda} \right)^{1/p}.$$

Proof.

$$\begin{aligned}
\|f\|^p_{L^p(X,\mu)} &= \int_X |f(x)|^p \, d\mu(x) \\
&= \int_X \int_0^{|f(x)|} p\lambda^p \, \tfrac{d\lambda}{\lambda} \, d\mu(x) \\
&= p \int_X \int_{\mathbb{R}^+} \lambda^p \mathbb{1}_{\{s:\, s<|f(x)|\}} \, \tfrac{d\lambda}{\lambda} \, d\mu(x) \\
&= p \int_{\mathbb{R}^+} \lambda^p \int_X \mathbb{1}_{\{y:|f(y)|>\lambda\}} \, d\mu(x) \, \tfrac{d\lambda}{\lambda} && \text{(Fubini)} \\
&= p \int_{\mathbb{R}^+} \lambda^p d_f(\lambda) \, \tfrac{d\lambda}{\lambda}.
\end{aligned}$$

□

So we have:

$$\|f\|_{L^{p,\infty}} = \|\lambda d_f(\lambda)^{1/p}\|_{L^\infty(\mathbb{R}^+, \frac{d\lambda}{\lambda})}$$

and

$$\|f\|_{L^p} = p^{1/p} \|\lambda d_f(\lambda)^{1/p}\|_{L^p(\mathbb{R}^+, \frac{d\lambda}{\lambda})}.$$

From this, we could tentatively define the

Definition 1.7 (Lorentz quasi-norm)**.** Let (X, μ) be a measure space and let $0 < p < \infty$ and $0 < q \leq \infty$, we define the Lorentz space quasi-norm $\| \cdot \|_{L^{p,q}(X,\mu)}$ as

$$\|f\|_{L^{p,q}(X,\mu)} = p^{1/q} \|\lambda d_f(\lambda)^{1/p}\|_{L^q\left(\mathbb{R}^+, \frac{d\lambda}{\lambda}\right)}.$$

Observe that, with this definition, $L^{p,p}$ (isometrically) coincides with L^p.

We have the following useful lemma, which is a kind of Monotone Convergence Theorem.

Lemma 1.8. *Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of measurable functions such that $|f_n| \nearrow |f|$ μ-almost everywhere, then*

$$\|f\|_{L^{p,q}} = \lim_{n\to\infty} \|f_n\|_{L^{p,q}}$$

Proof. We can obviously assume f to be positive so, by definition, we only have to verify that

$$\lim_{n\geq 1} \|\lambda d_{f_n}(\lambda)^{1/p}\|_{L^q(\mathbb{R}^+, \frac{d\lambda}{\lambda})} = \|\lambda d_f(\lambda)^{1/p}\|_{L^q(\mathbb{R}^+, \frac{d\lambda}{\lambda})}.$$

Note that, since f_n is non-decreasing, $d_{f_n}(\cdot)$ is also increasing (in n), so in particular by Lemma 1.3

$$d_f(\lambda) = \sup_{n\geq 1} d_{f_n}(\lambda)$$

for all fixed $\lambda > 0$.

If $q = \infty$ then the problem is reduced to the easier one of showing that

$$\sup_{n\geq 1} \|\lambda d_{f_n}(\lambda)^{1/p}\|_{L^\infty_\lambda(\mathbb{R}^+)} = \|\sup_{n\geq 1} \lambda d_{f_n}(\lambda)^{1/p}\|_{L^\infty_\lambda(\mathbb{R}^+)},$$

which is easily seen to hold true in the more general case

$$\sup_{n\geq 1} \|g_n\|_{L^\infty} = \|\sup_{n\geq 1} g_n\|_{L^\infty}.$$

Suppose now that $q < \infty$, then

$$\sup_{n\geq 1} \|\lambda d_{f_n}(\lambda)^{1/p}\|_{L^q(\mathbb{R}^+,\frac{d\lambda}{\lambda})} = \left(\sup_{n\geq 1} \int_0^\infty \lambda^q d_{f_n}(\lambda)^{q/p} \,\tfrac{d\lambda}{\lambda}\right)^{1/q},$$

and since, as we noted before, we have $\lambda^q d_{f_n}(\lambda)^{q/p} \nearrow \lambda^q d_f(\lambda)^{q/p}$ for all fixed $\lambda > 0$, we can apply the monotone convergence theorem with the measure $\frac{d\lambda}{\lambda}$ and the desired identity. $\square$

The following Fatou-like lemma shows that the Lorentz space quasi-norm is lower semi-continuous.

Lemma 1.9. *Let $\{f_n\}_{n\in\mathbb{N}}$ be any sequence of measurable functions, then*

$$\|\liminf_n f_n\|_{L^{p,q}} \leq \liminf_n \|f_n\|_{L^{p,q}}$$

Proof. Let $f = \liminf_n f_n$, then, by definition, we have to check that

$$\|\lambda d_f(\lambda)^{1/p}\|_{L^q(\mathbb{R}^+,\frac{d\lambda}{\lambda})} \leq \liminf_n \|\lambda d_{f_n}(\lambda)^{1/p}\|_{L^q(\mathbb{R}^+,\frac{d\lambda}{\lambda})},$$

which would follow from the lower-semicontinuity of the usual L^q spaces provided that we have

$$d_f(\lambda) \leq \liminf_n d_{f_n}(\lambda).$$

This is equivalent to showing

$$\mu(\{x : \liminf_n |f_n(x)| > \lambda\}) \leq \liminf_n \mu(\{x : |f_n(x)| > \lambda\}),$$

which follows from Fatou's lemma. $\square$

Finally, we can prove that the $L^{p,q}$ spaces are quasi-Banach spaces; which follows the steps of the classical proof in the L^p case.

Theorem 1.10. *Let (X, μ) be a measure space and let $0 < p < \infty$, $0 < q \leq \infty$, then $L^{p,q}(X, \mu)$ is a quasi-Banach space, that is, it is complete and it satisfies the quasi-triangle inequality.*

Proof. We have already proven that $L^{p,\infty}$ satisfies the quasi-triangle inequality in Proposition 1.5, the proof for $L^{p,q}$ when $q \neq \infty$ is almost identical so we will omit it.

We are left to prove that it is complete. To this end first note that the $L^{p,q}$ norm always controls the $L^{p,\infty}$ norm:

$$\begin{aligned}
\|\lambda d_f(\lambda)^{1/p}\|_{L^q(\mathbb{R}^+,\frac{d\lambda}{\lambda})} &= \left(\int_0^\infty \lambda^q d_f(\lambda)^{q/p} \,\tfrac{d\lambda}{\lambda}\right)^{1/q} \\
&\geq \left(\int_0^s \lambda^q d_f(\lambda)^{q/p} \,\tfrac{d\lambda}{\lambda}\right)^{1/q} \\
&\geq d_f(s)^{1/p} \left(\int_0^s \lambda^q \,\tfrac{d\lambda}{\lambda}\right)^{1/q} \\
&= s d_f(s)^{1/p} q^{-1/q},
\end{aligned}$$

for any $s > 0$. Thus, taking the supremum on s, we obtain:

$$\|f\|_{L^{p,\infty}} \lesssim_{p,q} \|f\|_{L^{p,q}}.$$

In particular we have the following generalization of Chebychev's inequality:

$$d_f(\lambda) \lesssim_{p,q} \frac{\|f\|_{L^{p,q}}^p}{\lambda^p}. \tag{4}$$

Now let $\{f_n\}$ be a Cauchy sequence in the $L^{p,q}$ quasi-norm. Estimate (4) allows us to deduce that $\{f_n\}$ is Cauchy in measure, and hence that there exists a μ-almost everywhere convergent subsequence $\{f_{\varphi(k)}\}_k$. Call f the μ-almost everywhere defined limit of $f_{\varphi(k)}$ when $k \to \infty$, we will show that $f_n \to f$ in the $L^{p,q}$ quasi-norm.

Let C be the constant in the quasi-triangle inequality for the $\|\cdot\|_{L^{p,q}}$ quasi-norm and refine the subsequence $\{f_{\varphi(k)}\}$ so that we have

$$\|f_{\varphi(k)} - f_{\varphi(k-1)}\|_{L^{p,q}} \le (2C)^{-k}$$

using that f_n is an $L^{p,q}$-Cauchy sequence and defining $f_{\varphi(0)} = 0$. Observe that we have the following telescoping sum identity:

$$f_{\varphi(n)} = \sum_{k=1}^{n} f_{\varphi(k)} - f_{\varphi(k-1)}. \tag{5}$$

Recall that we want to estimate $\|f - f_n\|_{L^{p,q}}$ and that, using the quasi-triangle inequality, we have

$$\|f - f_n\|_{L^{p,q}} \le C\left(\|f - f_{\varphi(N)}\|_{L^{p,q}} + \|f_{\varphi(N)} - f_n\|_{L^{p,q}}\right).$$

The second term can be made arbitrarily small using the fact that f_n is an $L^{p,q}$-Cauchy sequence, so we choose n and N so large that

$$\|f_{\varphi(N)} - f_n\|_{L^{p,q}} < \frac{\epsilon}{2C}.$$

For the first term we can use (5) to obtain:

$$\begin{aligned}
\|f - f_{\varphi(N)}\|_{L^{p,q}} &= \left\|\sum_{k=1}^{\infty}(f_{\varphi(k)} - f_{\varphi(k-1)}) - \sum_{k=1}^{N}(f_{\varphi(k)} - f_{\varphi(k-1)})\right\|_{L^p,q} \\
&= \left\|\lim_{M\to\infty}\sum_{k=N+1}^{M}\left(f_{\varphi(k)} - f_{\varphi(k-1)}\right)\right\|_{L^{p,q}} \\
&\le \left\|\lim_{M\to\infty}\sum_{k=N+1}^{M}|f_{\varphi(k)} - f_{\varphi(k-1)}|\right\|_{L^{p,q}} \\
&\le \liminf_{M\to\infty}\left\|\sum_{k=N+1}^{M}|f_{\varphi(k)} - f_{\varphi(k-1)}|\right\|_{L^{p,q}}, \qquad (6)
\end{aligned}$$

where in the last line we have used Proposition 1.9. By induction, we can iteratively use the quasi-triangle inequality to exchange $\sum$ and $\|\cdot\|_{L^{p,q}}$ and bound the last term in (6) by

$$\begin{aligned}
\liminf_{M\to\infty}\sum_{k=N+1}^{M} C^{k-N}\|f_{\varphi(k)} - f_{\varphi(k-1)}\|_{L^{p,q}} &\le \liminf_{M\to\infty}\sum_{k=N+1}^{M} C^{k-N}(2C)^{-k} \\
&= C^{-N}\sum_{k=N+1}^{\infty} 2^{-k} \\
&\le (2C)^{-N}.
\end{aligned}$$

Choosing N so large that $\frac{1}{(2C)^N} < \frac{\epsilon}{2C}$ completes the proof. □

2. The Dyadic Decomposition

It turns out that we will be able to decompose functions in $L^{p,q}$ into simpler functions:

- Sub-step functions: A sub-step function of height H and width W is any measurable function f supported on a set of measure W and which satisfies $|f| \le H$.

- Quasi-step function: A quasi-step function of height H and width W is any measurable function f supported on a set of measure W and which satisfies $|f| \sim H$ in its support.

Example 2.1. Let f be a quasi-step function of height H and width W, then

$$\|f\|_{L^{p,q}} \sim HW^{1/p} \left(\frac{p}{q}\right)^{1/q}.$$

We will give two different decompositions: a "vertically-dyadic" decomposition; where we decompose the range of our function into dyadic pieces, and a "horizontally-dyadic" one; where we dyadically decompose its support.

Let $\{\gamma_m\}_{m\in\mathbb{Z}}$ be a sequence in $\mathbb{R}^+$, we say that it is of *lower exponential growth* if we have

$$C_1\gamma_m \leq \gamma_{m+1}$$

for some constant $C_1 > 1$. We will say that it is of *exponential growth* if we additionally have

$$\gamma_{m+1} \leq C_2\gamma_m$$

for some $C_2 > 1$.

Note that the condition of being of *lower exponential growth* can easily be tested by the equivalent condition of

$$\inf_{m\in\mathbb{Z}} \frac{\gamma_{m+1}}{\gamma_m} > 1,$$

while that of being of *exponential growth* is in turn equivalent to being of lower exponential growth and

$$\sup_{m\in\mathbb{Z}} \frac{\gamma_{m+1}}{\gamma_m} < \infty.$$

We can now state the main theorem of this section:

Theorem 2.2. *Let $f \geq 0$ be a measurable function, $\{\gamma_m\}_{m\in\mathbb{Z}}$ be a sequence of exponential growth and $0 < p < \infty$, $1 \leq q \leq \infty$. Then the following statements are equivalent:*

(a) *There exists a decomposition $f = \sum_{m\in\mathbb{Z}} f_m$, where each f_m is a quasi-step function of height γ_m and width W_m, the f_m's have pairwise disjoint supports and we have*

$$\left\|\gamma_m W_m^{1/p}\right\|_{\ell^q} \lesssim_{p,q} 1. \tag{7}$$

(b) *We have the estimate*

$$\|f\|_{L^{p,q}} \lesssim_{p,q} 1.$$

(c) *There exists a decomposition $f = \sum_{m\in\mathbb{Z}} f_m$, where each f_m is a sub-step function of width γ_m and height H_m, the f_m's have pairwise disjoint supports, H_m is non-increasing, $H_{m+1} \leq |f_m| \leq H_m$ in the support of f_m and we have*

$$\left\|H_m\gamma_m^{1/p}\right\|_{\ell^q} \lesssim_{p,q} 1. \tag{8}$$

Proof. Observe that, since $\{\gamma_m\}_{\mathbb{Z}}$ is of lower exponential growth, we have

$$\gamma_{m-k} \leq C_1^{-k}\gamma_m \quad \text{and} \quad \gamma_{m+k} \geq C_1^k\gamma_m \tag{9}$$

for all $m \in \mathbb{Z}$ and $k \geq 0$. This implies in particular that $\gamma_m \to 0$ when $m \to -\infty$ and $\gamma_m \to \infty$ when $m \to \infty$.

Let us first prove that (a) $\implies$ (b): By monotonicity we can assume that

$$f = \sum_{m\in\mathbb{Z}} \gamma_m \mathbb{1}_{E_m}$$

where E_m is the support of f_m. Since the sequence γ_m is (strictly) increasing, we have

$$d_f(\gamma_m) = \sum_{k=1}^{\infty} \mu(E_{m+k}).$$

Let us first suppose that $q = \infty$, we have to prove that

$$\sup_{\lambda>0} \lambda d_f(\lambda)^{1/p} \lesssim_p 1.$$

To this end observe that

$$\begin{aligned}
\sup_{\lambda>0} \lambda d_f(\lambda)^{1/p} &= \sup_{m\in\mathbb{Z}} \sup_{\lambda\in(\gamma_m,\gamma_{m+1}]} \lambda d_f(\lambda)^{1/p} \\
&\leq \sup_{m\in\mathbb{Z}} \gamma_{m+1} d_f(\gamma_m)^{1/p} \\
&= \sup_{m} \gamma_{m+1} \left(\sum_{k=1}^{\infty} W_{m+k}\right)^{1/p} \\
&\leq \left(\sum_{k=1}^{\infty} \sup_{m\in\mathbb{Z}} \gamma_{m+1}^p W_{m+k}\right)^{1/p} \\
&= \left(\sum_{k=1}^{\infty} \sup_{m\in\mathbb{Z}} \gamma_{m+1-k}^p W_{m}\right)^{1/p} \\
&\leq \left(\sup_{m\in\mathbb{Z}} \gamma_m^p W_m \sum_{k=1}^{\infty} C_1^{-p(k-1)}\right)^{1/p} \qquad (9) \\
&\lesssim_p \sup_{m\in\mathbb{Z}} \gamma_m W_m^{1/p} \\
&\lesssim 1.
\end{aligned}$$

Now suppose $q < \infty$, then we can do something similar:

$$\begin{aligned}
\|f\|_{L^{p,q}}^q &\sim_{p,q} \|\lambda d_f(\lambda)^{1/p}\|_{L^q(\mathbb{R}^+,\frac{d\lambda}{\lambda})}^q \\
&= \sum_{m\in\mathbb{Z}} \int_{\gamma_m}^{\gamma_{m+1}} \lambda^q d_f(\lambda)^{q/p} \, \frac{d\lambda}{\lambda} \\
&\leq \sum_{m\in\mathbb{Z}} d_f(\gamma_m)^{q/p} \int_{\gamma_m}^{\gamma_{m+1}} \lambda^q \, \frac{d\lambda}{\lambda} \\
&\lesssim_q \sum_{m\in\mathbb{Z}} d_f(\gamma_m)^{q/p} \gamma_{m+1}^q \\
&= \left\| d_f(\gamma_m) \gamma_{m+1}^p \right\|_{\ell^{q/p}}^{q/p} \\
&= \left\| \sum_{k=1}^{\infty} \gamma_{m+1}^p W_{m+k} \right\|_{\ell^{q/p}}^{q/p}.
\end{aligned}$$

We would like to exchange $\|\cdot\|_{\ell^{p/q}}$ with $\sum$, but this is not possible in general unless $q/p \geq 1$. We could fix this using the q/p-convexity of $\ell^{q/p}$ for $0 < q/p < 1$, but we will treat it in a unified way using Lemma 5.1:

From (9) and (7), we have:

$$\begin{aligned}\|\gamma_{m+1}^p W_{m+k}\|_{\ell_m^{q/p}} &= \|\gamma_{m-(k-1)}^p W_m\|_{\ell_m^{q/p}} \\ &\le C_1^{-p(k-1)} \|\gamma_m^p W_m\|_{\ell_m^{q/p}} \\ &\lesssim_{C_1} C_1^{-pk} \|\gamma_m^p W_m\|_{\ell_m^{q/p}} \\ &= C_1^{-pk} \|\gamma_m W_m^{1/p}\|_{\ell_m^q}^p \\ &\lesssim_p C_1^{-pk}.\end{aligned}$$

Hence, using Lemma 5.1, we obtain:

$$\left\| \sum_{k=1}^{\infty} \gamma_{m+1}^p W_{m+k} \right\|_{\ell^{q/p}}^{q/p} \lesssim_{C_1,p,q} 1.$$

Let us now see that (b) $\implies$ (c):
Define

$$H_m = \inf\{\lambda > 0 \, : \, d_f(\lambda) \le \gamma_m\},$$

Clearly H_m is non-increasing and, by definition, we have

$$\lambda < H_m \implies d_f(\lambda) > \gamma_m.$$

Observe also that, from the monotone convergence theorem, this infumum is achieved and so $d_f(H_m) = \gamma_m$. Since $\|f\|_{L^{p,q}}$ is finite, $d_f(\lambda)$ has to be finite for $\lambda > 0$ (otherwise $d_f(s)$ would be infinite for $0 < s < \lambda$). Thus, if $H_m \neq 0$, then the sets

$$E_m = \{x : f(x) > H_m\}$$

are of finite measure. Using the monotone convergence theorem again we conclude that the set

$$E = \{x : f(x) \ge \sup_m H_m\}$$

has measure 0. By the definition of H_m

$$d_f(\lambda) \le \gamma_m \implies H_m \le \lambda,$$

thus if $d_f(\epsilon) \le \gamma_m$ (which is true for m sufficiently large since $\gamma_m \to \infty$ when $m \to \infty$), then $H_m \le \epsilon$, that is $H_m \to 0$ when $m \to \infty$.

These remarks show that

$$f = \sum_{m \in \mathbb{Z}} f_m,$$

where $f_m = f \mathbb{1}_{H_{m+1} < f \le H_m}$.

If $q < \infty$ then

$$\begin{aligned}\|\lambda d_f(\lambda)^{1/p}\|_{L^q((H_{m+k+1}, H_{m+k}], \frac{d\lambda}{\lambda})}^q &= \int_{H_{m+k+1}}^{H_{m+k}} \lambda^q d_f(\lambda)^{q/p} \, \tfrac{d\lambda}{\lambda} \\ &\ge \gamma_{m+k}^{q/p} \int_{H_{m+k+1}}^{H_{m+k}} \lambda^q \, \tfrac{d\lambda}{\lambda} \\ &\sim_q \gamma_{m+k}^{q/p} \left(H_{m+k}^q - H_{m+k+1}^q \right) \\ &\gtrsim_{C_1,p,q} C_1^{kq/p} \gamma_m^{q/p} \left(H_{m+k}^q - H_{m+k+1}^q \right).\end{aligned}$$

Since $H_m \to 0$ when $m \to \infty$, we can sum telescopically as follows:

$$\begin{aligned} H_m \gamma_m^{1/p} &= \left(H_m^q \gamma_m^{q/p}\right)^{1/q} \\ &= \left(\sum_{k=0}^{\infty} (H_{m+k}^q - H_{m+k+1}^q)\gamma_m^{q/p}\right)^{1/q} \\ &\lesssim_{C_1,p,q} \left(\sum_{k=0}^{\infty} C_1^{-kq/p} \|\lambda d_f(\lambda)^{1/p}\|^q_{L^q((H_{m+k+1},H_{m+k}],\frac{d\lambda}{\lambda})}\right)^{1/q}, \end{aligned}$$

so taking ℓ^q norms

$$\begin{aligned} \|H_m\gamma_m^{1/p}\|^q_{\ell^q} &\lesssim_{C_1,p,q} \sum_{m\in\mathbb{Z}}\sum_{k=0}^{\infty} C_1^{-kq/p}\|\lambda d_f(\lambda)^{1/p}\|^q_{L^q((H_{m+k+1},H_{m+k}],\frac{d\lambda}{\lambda})} \\ &= \sum_{k=0}^{\infty} C_1^{-kq/p} \sum_{m\in\mathbb{Z}} \|\lambda d_f(\lambda)^{1/p}\|^q_{L^q((H_{m+k+1},H_{m+k}],\frac{d\lambda}{\lambda})} \\ &= \sum_{k=0}^{\infty} C_1^{-kq/p}\|\lambda d_f(\lambda)^{1/p}\|^q_{L^q(\mathbb{R}^+,\frac{d\lambda}{\lambda})} \\ &\lesssim_{C_1,p,q} 1. \end{aligned}$$

Lastly, let $q = \infty$. Then we immediately have

$$d_f(\lambda) \lesssim \lambda^{-p},$$

so if $\lambda > \gamma_m^{-1/p}$, then $d_f(\lambda) \lesssim \gamma_m$, that is: $H_m \lesssim \gamma_m^{-1/p}$. This implies

$$\|H_m\gamma_m^{1/p}\|_{\ell^\infty} \lesssim 1,$$

which is what we wanted.

We now prove the implications in the reverse order. Let's start with $(c) \implies (b)$: By hypothesis and the estimate (9) we have

$$d_f(H_m) \le \sum_{k=1}^{\infty} \gamma_{m-k} \le \gamma_m \sum_{k=1}^{\infty} C_1^{-k} \lesssim_{C_1} \gamma_m.$$

Observe also that it follows from (7) that $H_m \to 0$ when $m \to \infty$. Additionally:

$$\lambda < \sup_{m\in\mathbb{Z}} H_m \implies \lambda \in \bigcup_{m\in\mathbb{Z}} [H_{m+1}, H_m).$$

So, if $q < \infty$:

$$\begin{aligned} \|\lambda d_f(\lambda)^{1/p}\|^q_{L^q(\mathbb{R}^+,\frac{d\lambda}{\lambda})} &= \sum_{m\in\mathbb{Z}} \int_{H_{m+1}}^{H_m} \lambda^q d_f(\lambda)^{q/p}\,\frac{d\lambda}{\lambda} \\ &\lesssim_{C_1,q} \sum_{m\in\mathbb{Z}} \gamma_{m+1}^{q/p}\left(H_m^q - H_{m+1}^q\right) \\ &\lesssim_{C_2,p} \sum_{m\in\mathbb{Z}} \gamma_m^{q/p} H_m \\ &= \|\gamma_m^{1/p}H_m\|^q_{\ell^q} \\ &\lesssim 1. \end{aligned}$$

If $q = \infty$ then we can similarly estimate:

$$\begin{aligned}\sup_{\lambda>0} \lambda d_f(\lambda)^{1/p} &= \sup_{m\in\mathbb{Z}} \sup_{\lambda\in[H_{m+1},H_m)} \lambda d_f(\lambda)^{1/p} \\ &\leq \sup_{m\in\mathbb{Z}} H_m d_f(H_{m+1})^{1/p} \\ &\lesssim_{C_1,p} \sup_{m\in\mathbb{Z}} H_m \gamma_{m+1}^{1/p} \\ &\lesssim_{C_2,p} \sup_{m\in\mathbb{Z}} H_m \gamma_m^{1/p} \\ &\lesssim 1.\end{aligned}$$

The final stage of the proof is showing that

$$(b) \implies (a).$$

Define

$$f_m = f \mathbb{1}_{\gamma_m < |f| \leq \gamma_{m+1}},$$

then clearly $W_m = d_f(\gamma_m) - d_f(\gamma_{m+1}) \leq d_f(\gamma_m)$.

If $q < \infty$ we proceed as follows:

$$\begin{aligned}1 &\gtrsim \|\lambda d_f(\lambda)^{1/p}\|^q_{L^q(\mathbb{R}^+,\frac{d\lambda}{\lambda})} \\ &= \sum_{m\in\mathbb{Z}} \int_{\gamma_m}^{\gamma_{m+1}} \lambda^q d_f(\lambda)^{q/p} \tfrac{d\lambda}{\lambda} \\ &\geq \sum_{m\in\mathbb{Z}} d_f(\gamma_m)^{q/p} \left(\gamma_{m+1}^q - \gamma_m^q\right) \\ &\gtrsim_{C_2} \sum_{m\in\mathbb{Z}} d_f(\gamma_m)^{q/p} \gamma_m^q \\ &\geq \sum_{m\in\mathbb{Z}} W_m^{q/p} \gamma_q \\ &= \left\|\gamma_m W_m^{1/p}\right\|^q_{\ell^q}.\end{aligned}$$

If $q = \infty$, then we have to show that

$$\sup_m \gamma_m W_m^{1/p} \lesssim 1,$$

but by hypothesis $d_f(\lambda)^{1/p} \lesssim \lambda^{-1}$, so

$$W_m^{1/p} \leq d_f(\gamma_m)^{1/p} \lesssim \gamma_m^{-1}$$

and the proof follows. □

Remark 2.3. Observe that the proof of (a) $\implies$ (b) works verbatim when the sequence $\{\gamma_m\}_{m\in\mathbb{Z}}$ is only of lower exponential growth.

3. Quantitative Properties of $L^{p,q}$ spaces

As a first application of the dyadic decomposition theorems of the last section we can prove the following generalization of Hölder's inequality, originally proven by O'Neil in [4].

Theorem 3.1. *Let* $0 < p, p_1, p_2 < \infty$ *and* $0 < q, q_1, q_2 \leq \infty$ *exponents satisfying*

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \quad \textit{and} \quad \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2},$$

then for every measurable function we have

$$\|fg\|_{L^{p,q}} \lesssim_{p_1,q_1,p_2,q_2} \|f\|_{L^{p_1,q_1}} \|g\|_{L^{p_2,q_2}}.$$

Proof. First observe that we can assume $\|f\|_{L^{p_1,q_1}} = \|g\|_{L^{p_2,q_2}} = 1$ and that $f, g \geq 0$, so we only have to prove

$$\|fg\|_{L^{p,q}} \lesssim 1.$$

We will omit the dependence on the exponents for the rest of the proof.

Decompose f and g using the horizontally dyadic decomposition with $\gamma_m = 2^m$, i.e.:

$$f = \sum_m f_m, \quad g = \sum_m g_m$$

where f_m and g_m are sub-step functions of height H_m and H'_m respectively and have supports of measure $\leq 2^m$.

To estimate the fg we proceed as follows:

$$\begin{aligned} fg &\leq \sum_m \sum_n f_m g_n \\ &= \sum_k \sum_m f_m g_{m+k} \\ &= \sum_{k<0} \sum_m f_m g_{m+k} + \sum_{k\geq 0} \sum_m f_m g_{m+k}, \end{aligned}$$

since the $L^{p,q}$ quasi-norm satisfies the quasi-triangle inequality it suffices to estimate the $k < 0$ and $k \geq 0$ sums separately. We will only show

$$\|\sum_{k\geq 0} \sum_m f_m g_{m+k}\|_{L^{p,q}} \lesssim 1$$

since the other sum is estimated in the same way.

Observe that $f_m g_{m+k}$ is a sub-step function of height $H_m H'_{m+k}$ and width $\leq \min(2^m, 2^{m+k})$, so

$$\sum_m f_m g_{m+k} \leq \sum_m H_m H'_{m+k} \mathbb{1}_{E_m},$$

where E_m is a set of measure 2^m. We can now employ the implication $(c) \implies (b)$ in Theorem 2.2 to obtain:

$$\begin{aligned} \|\sum_m f_m g_{m+k}\|_{L^{p,q}} &\lesssim \left\|H_m H'_{m+k} 2^{m/p}\right\|_{\ell^q} \\ &= \left\|2^{m/p_1} H_m 2^{m/p_2} H'_{m+k}\right\|_{\ell^q} \\ &\leq \left\|2^{m/p_1} H_m\right\|_{\ell^{q_1}} \left\|2^{m/p_2} H'_{m+k}\right\|_{\ell^{q_2}} \\ &\lesssim 2^{-k/p_2}, \end{aligned}$$

where we have used Hölder's inequality in the last line.

To get

$$\left\|\sum_{k\geq 0} \sum_m f_m g_{m+k}\right\|_{L^{p,q}} \lesssim 1$$

we just use Lemma 5.1. □

The following dual characterization of the $L^{p,\infty}$ space will be useful.

Theorem 3.2. *Let (X, μ) be a σ-finite measure space, $1 < p < \infty$ and $f \in L^{p,\infty}$. Then*

(i) *If* $f \in L^{p,\infty}$ *then*

$$\|f\|_{L^{p,\infty}} \sim_p \sup_{0<\mu(E)<\infty} \frac{1}{\mu(E)^{1/p'}} \left| \int_X f \mathbb{1}_E \, d\mu \right|. \tag{10}$$

(ii) *If* $f \mathbb{1}_E$ *is integrable for all sets* E *of finite measure and*

$$M_p = \sup_{0<\mu(E)<\infty} \frac{1}{\mu(E)^{1/p'}} \left| \int_X f \mathbb{1}_E \, d\mu \right| < \infty,$$

then $f \in L^{p,\infty}$ *and* $\|f\|_{L^{p,\infty}} \lesssim_p M_p$.

Proof. Let's prove (i) first. The $\gtrsim_p$ part is a simple consequence of Hölder's inequality for Lorentz spaces, so we will concentrate on showing the reverse inequality.

Let us first assume f to be non-negative. Let $E_\lambda = \{x \in X : f(x) > \lambda\}$, the assumption $f \in L^{p,\infty}$ guarantees that $\mu(E_\lambda) < \infty$. Then, if $\mu(E_\lambda) \neq 0$:

$$\lambda \mu(E_\lambda)^{1/p} = \frac{1}{\mu(E_\lambda)^{1/p'}} \lambda \mu(E_\lambda) \leq \frac{1}{\mu(E_\lambda)^{1/p'}} \int_X f \mathbb{1}_{E_\lambda} \, d\mu \leq \sup_{0<\mu(E)<\infty} \frac{1}{\mu(E)^{1/p'}} \left| \int_X f \mathbb{1}_E \, d\mu \right|,$$

taking the supremum in $\lambda > 0$ yields (10).

Now suppose f is not necessarily non-negative. We can decompose $f = u_0^+ - u_0^- + i(u_1^+ - u_1^-)$ where $u_i^\pm \geq 0$ for $i \in \{0, 1\}$, so by the quasi-triangle inequality:

$$\|f\|_{L^{p,\infty}} \lesssim_p \|u_0^+\|_{L^{p,\infty}} + \|u_0^-\|_{L^{p,\infty}} + \|u_1^+\|_{L^{p,\infty}} + \|u_1^-\|_{L^{p,\infty}}.$$

It thus suffices to show that

$$\|u_i^\pm\|_{L^{p,\infty}} \leq \sup_{0<\mu(E)<\infty} \frac{1}{\mu(E)^{1/p'}} \left| \int_X f \mathbb{1}_E \, d\mu \right|,$$

but since $u_i^\pm$ is non-negative, we can use what we already know to reduce the problem to showing that

$$\sup_{0<\mu(E)<\infty} \frac{1}{\mu(E)^{1/p'}} \left| \int_X u_i^\pm \mathbb{1}_E \, d\mu \right| \leq \sup_{0<\mu(E)<\infty} \frac{1}{\mu(E)^{1/p'}} \left| \int_X f \mathbb{1}_E \, d\mu \right|.$$

Finally, observe that $u_i^\pm = e^{i\theta} f \mathbb{1}_F$ for some $F \subseteq X$, so for any given set E with $0 < \mu(E) < \infty$ we have that

$$\frac{1}{\mu(E)^{1/p'}} \left| \int_X u_i^\pm \mathbb{1}_E \, d\mu \right| = 0 \leq \sup_{0<\mu(E')<\infty} \frac{1}{\mu(E')^{1/p'}} \left| \int_X f \mathbb{1}_{E'} \, d\mu \right|$$

if $\mu(E \cap F) = 0$ and

$$\begin{aligned} \frac{1}{\mu(E)^{1/p'}} \left| \int_X u_i^\pm \mathbb{1}_E \, d\mu \right| &= \frac{1}{\mu(E)^{1/p'}} \left| \int_X e^{i\theta} f \mathbb{1}_{E\cap F} \, d\mu \right| \\ &\leq \frac{1}{\mu(E \cap F)^{1/p'}} \left| \int_X f \mathbb{1}_{E\cap F} \, d\mu \right| \\ &\leq \sup_{0<\mu(E')<\infty} \frac{1}{\mu(E')^{1/p'}} \left| \int_X f \mathbb{1}_{E'} \, d\mu \right| \end{aligned}$$

if $\mu(E \cap F) > 0$. Taking the supremum over these sets yields the proof.

Let us now prove (ii). The idea is to approximate f by functions f_n which are a priori in $L^{p,\infty}$, use (i) on these functions and then try to pass to the limit.

Indeed, since (X, μ) is σ-finite we can find an increasing sequence of sets $\{F_n\}_{n\in\mathbb{N}}$ of finite measure and such that

$$X = \bigcup_n F_n.$$

We now define $f_n = f\mathbb{1}_{E_n}$ with $E_n = F_n \cap \{x : |f(x)| \leq n\}$. These functions are qualitatively in $L^{p,\infty}$:

$$\|f_n\|_{L^{p,\infty}} \lesssim n\mu(E_n)^{1/p},$$

and also $|f_n| \nearrow |f|$, so we can use Proposition 1.9 to get:

$$\|f\|_{L^{p,\infty}} \leq \liminf_{n\to\infty} \|f_n\|_{L^{p,\infty}}. \tag{11}$$

We would have finished if the right hand side of (11) were finite, because then we would have $f \in L^{p,\infty}$ a priori. But observe that we can use (i) to bound each of these functions:

$$\|f_n\|_{L^{p,\infty}} \lesssim_p \sup_{0<\mu(E)<\infty} \frac{1}{\mu(E)^{1/p'}} \left| \int_X f_n \mathbb{1}_E \, d\mu \right|.$$

Now we use a procedure similar to that in the proof of (i):

$$\begin{aligned}
\sup_{0<\mu(E)<\infty} \frac{1}{\mu(E)^{1/p'}} \left| \int_X f \mathbb{1}_{E\cap E_n} \, d\mu \right| &\leq \sup_{0<\mu(E)<\infty} \frac{1}{\mu(E\cap E_n)^{1/p'}} \left| \int_X f \mathbb{1}_{E\cap E_n} \, d\mu \right| \\
&= \sup_{\mu(E')>0,\, E'\subseteq E_n} \frac{1}{\mu(E')^{1/p'}} \left| \int_X f \mathbb{1}_{E'} \, d\mu \right| \\
&\leq \sup_{0<\mu(E')<\infty} \frac{1}{\mu(E')^{1/p'}} \left| \int_X f \mathbb{1}_{E'} \, d\mu \right| \\
&= M_p
\end{aligned}$$

□

We can also prove a duality theorem for $L^{p,q}$ spaces:

Theorem 3.3. *Let $1 < p < \infty$ and $1 \leq q \leq \infty$, then for all $f \in \Sigma^+$:*

$$\|f\|_{L^{p,q}} \sim_{p,q} \sup\left\{ \int_X fg \, d\mu \,:\, g \in \Sigma^+,\, \|g\|_{L^{p',q'}} \leq 1 \right\} =: M_{p,q}(f), \tag{12}$$

where Σ^+ is the set of all non-negative simple functions with support of finite measure.

Proof. As before, the $\gtrsim_{p,q}$ inequality follows from Hölder's inequality for Lorentz spaces. Also, if $q = \infty$, then the result follows from Theorem 3.2. We will thus prove the inequality $\lesssim_{p,q}$ restricted to $q < \infty$.

Homogeneity allows us to normalize $\|f\|_{L^{p,q}} = 1$, we thus have to show that

$$M_{p,q}(f) \gtrsim_{p,q} 1.$$

Using Theorem 2.2 we can decompose f as

$$f = \sum_{m\in\mathbb{Z}} f_m,$$

where each f_m is a quasi-step function of height 2^m and width W_m, and such that

$$\left\|2^m W_m^{1/p}\right\|_{\ell^q} \sim 1.$$

Observe that it follows from the proof of Theorem 2.2 that each f_m must also be a simple function and, also, that f_m is 0 for $|m|$ sufficiently large.

Define the function g by

$$g = \sum_{m\in\mathbb{Z}} g_m$$

where $g_m = a_m^r f_m^{p-1}$, $a_m = 2^m W_m^{1/p}$ and $r = \max(0, q-p)$. Clearly $g \in \Sigma^+$, and furthermore, since the $f_m{}'s$ have disjoint supports E_m:

$$\begin{aligned}\int_X fg\,d\mu &= \int_X \sum_m a_m^r f_m^p\,d\mu\\ &= \sum_m a_m^r \int_X f_m^p\,d\mu.\end{aligned}$$

The functions f_m are quasi-step functions of height 2^m and width W_m, thus

$$\int_X f_m^p\,d\mu \sim_p 2^{mp} W_m = a_m^p,$$

that is

$$\int_X fg\,d\mu \sim_p \sum_{m\in\mathbb{Z}} a_m^{r+p} \overset{(*)}{\leq} \|a_m\|_{\ell^q}^{r+p} \sim_{p,q} \|f\|_{L^{p,q}} = 1,$$

where in $(*)$ we have used that $\|\cdot\|_{\ell^{p_1}} \leq \|\cdot\|_{\ell^{p_2}}$ when $p_1 \geq p_2$ and that $r+p \geq q$. So we only have to show that

$$\|g\|_{L^{p',q'}} \leq 1.$$

Observe that

$$g_m \lesssim_{p,q} a_m^r 2^{m(p-1)} \mathbb{1}_{E_m}$$

with $\mu(E_m) = W_m$, so a direct application of Theorem 2.2 would yield the result. The sequence $a_m^r 2^{m(p-1)}$, however, can fail to be of lower exponential growth (at least a priori). We can fix this by defining

$$\gamma_m = 2^{m\frac{p-1}{2}} \sup_{k\leq m} a_k^r 2^{k\frac{p-1}{2}} = 2^{m(p-1)} \sup_{k\geq 0} a_{m-k}^r 2^{-k\frac{p-1}{2}}.$$

Obviously $\gamma_m \geq a_m^r 2^{m(p-1)}$, and furthermore:

$$\begin{aligned}\gamma_{m+1} &= 2^{\frac{p-1}{2}} 2^{m\frac{p-1}{2}} \sup_{k\leq m+1} a_k^r 2^{k\frac{p-1}{2}}\\ &\geq 2^{\frac{p-1}{2}} 2^{m\frac{p-1}{2}} \sup_{k\leq m} a_k^r 2^{k\frac{p-1}{2}}\\ &= 2^{\frac{p-1}{2}} \gamma_m,\end{aligned}$$

that is, $\{\gamma_m\}_{m\in\mathbb{Z}}$ is of sub-exponential growth with constant $C_1 = 2^{\frac{p-1}{2}} > 1$. Now we can use Theorem 2.2 and the remark that follows it to reduce the problem to showing

$$\big\|\gamma_m W_m^{1/p'}\big\|_{\ell^{q'}} \lesssim_{p,q} 1.$$

But $W_m = 2^{-mp} a_m^p$, so

$$\begin{aligned}\big\|\gamma_m W_m^{1/p'}\big\|_{\ell^{q'}} &= \big\|\gamma_m a_m^{p-1} 2^{-m(p-1)}\big\|_{\ell^{q'}}\\ &= \Big\|a_m^{p-1} 2^{-m(p-1)} 2^{m(p-1)} \sup_{k\geq 0} a_{m-k}^r 2^{-k\frac{p-1}{2}}\Big\|_{\ell^{q'}}\\ &= \Big\|\sup_{k\geq 0} a_m^{p-1} a_{m-k}^r 2^{-k\frac{p-1}{2}}\Big\|_{\ell^{q'}}\\ &\leq \Big\|\sum_{k\geq 0} a_m^{p-1} a_{m-k}^r 2^{-k\frac{p-1}{2}}\Big\|_{\ell^{q'}}\\ &\leq \sum_{k=0}^{\infty} 2^{-k\frac{p-1}{2}} \big\|a_m^{p-1} a_{m-k}^r\big\|_{\ell^{q'}}.\end{aligned}$$

If $q \le p \implies r = 0$, but then $q' \ge p' \implies \|\cdot\|_{\ell^{q'}} \le \|\cdot\|_{\ell^{p'}}$, so

$$\left\|a_m^{p-1}\right\|_{\ell^{q'}} \le \left\|a_m^{p-1}\right\|_{\ell^{p'}} = \|a_m\|_{\ell^p}^{p-1} \le \|a_m\|_{\ell^q}^{p-1} = \left\|2^m W_m^{1/p}\right\|_{\ell^q}^{p-1} \lesssim_{p,q} 1,$$

which completes the proof if $q \le p$.

If $q > p$, then $r = q - p$ and

$$\frac{p-1}{q} + \frac{q-p}{q} = \frac{1}{q'},$$

so using Hölder's inequality:

$$\begin{aligned}\left\|a_m^{p-1} a_{m-k}^{q-p}\right\|_{\ell^{q'}} &\le \left\|a_m^{p-1}\right\|_{\ell^{\frac{q}{p-1}}} \left\|a_{m-k}^{q-p}\right\|_{\ell^{\frac{q}{q-p}}} \\ &= \|a_m\|_{\ell^q}^{p-1} \|a_{m-k}\|_{\ell^q}^{q-p} \\ &\lesssim_{p,q} 1.\end{aligned}$$

This finishes the proof. □

We can now strengthen this theorem to eliminate the qualitative assumptions of being simple:

Corollary 3.4. *Let $1 < p < \infty$ and $1 \le q \le \infty$, then*

$$\|f\|_{L^{p,q}} \sim_{p,q} \sup\left\{\left|\int_X fg\,d\mu\right| : g \in \Sigma,\ \|g\|_{L^{p',q'}} \le 1\right\} \tag{13}$$

for all $f \in L^{p,q}$ and where Σ is the set of all finite combinations of characteristic functions of sets of finite measure. Furthermore, if the measure space is σ-finite and if f is not a priori in $L^{p,q}$ but the right hand side is well defined and finite, t hen the left hand side is also finite and we have (13).

Proof. We will only show this in the case $f \in L^{p,q}$ and $f \ge 0$, the complete proof follows the same pattern as in the proof of Theorem 3.2.

From Hölder's inequality we can reduce to showing only the $\lesssim_{p,q}$ inequality. Approximate (a.e.) f by an increasing sequence $\{f_n\}_{n\in\mathbb{N}}$ of positive simple functions of finite support (this can be achieved if the measure space is σ-finite or if f is a priori in $L^{p,q}$, we omit the details).

Using the Monotone Convergence Theorem for Lorentz spaces 1.8:

$$\begin{aligned}\|f\|_{L^{p,q}} &= \sup_n \|f_n\|_{L^{p,q}} \\ &\sim_{p,q} \sup_n \sup\left\{\int_X f_n g\,d\mu : g \in \Sigma^+,\ \|g\|_{L^{p',q'}} \le 1\right\} \\ &= \sup\left\{\sup_n \int_X f_n g\,d\mu : g \in \Sigma^+,\ \|g\|_{L^{p',q'}} \le 1\right\} \\ &= \sup\left\{\int_X fg\,d\mu : g \in \Sigma^+,\ \|g\|_{L^{p',q'}} \le 1\right\},\end{aligned}$$

where we have used the (usual) monotone convergence theorem in the last line. □

This shows that there is a norm (the one defined using the dual characterization) which is equivalent to the $L^{p,q}$ quasi-norm defined at the beginning when $1 < p < \infty$.

4. Marcinkiewicz Interpolation Theorem

In this section we will present a proof of the Marcinkiewicz interpolation theorem using the decomposition introduced in section 2.

We will use duality from the outset, indeed, we will need the following extension of Theorem 3.2:

Theorem 4.1. *Let (X, μ) be a measure space and let $0 < p < \infty$, the following statements are equivalent for $f \in L^{p,\infty}$:*

(1) $\|f\|_{L^{p,\infty}} \lesssim_p 1$.

(2) *For every set E of finite measure there exists a subset $E' \subseteq E$ such that $\mu(E) \geq \frac{1}{2}\mu(E)$ such that*

$$\left| \int_X f \mathbb{1}_{E'} \, d\mu \right| \lesssim_p \mu(E')^{1/p'}.$$

Furthermore, the theorem holds without the a priori assumption of $f \in L^{p,\infty}$ provided (X, μ) is σ-finite.

Proof. We will assume $0 < p \leq 1$, we will also assume $f \geq 0$ and $\|f\|_{L^{p,\infty}} = 1$ without loss of generality. The proof of (2) $\implies$ (1) is almost identical to the one in Theorem 3.2, let us show the other one.

Let $E \subset X$ be a set of finite (and positive) measure and let $\alpha = \left(\frac{2}{\mu(E)}\right)^{1/p}$. Define $E' = E \setminus \{x : f(x) > \alpha\}$, then, since we have the estimate

$$\mu(\{x : f(x) > \alpha\}) \leq \frac{\mu(E)}{2},$$

we can bound $\mu(E) \geq \mu(E') \geq \frac{1}{2}\mu(E)$. Now we can estimate the integral

$$\begin{aligned} \int_X f \mathbb{1}_{E'} \, d\mu &\leq \left(\frac{2}{\mu(E)}\right)^{1/p} \mu(E') \\ &\leq 2^{1/p} \mu(E')^{1/p'}. \end{aligned}$$

□

The Marcinkiewicz interpolation theorem, as in complex interpolation, uses the boundedness of an operator in two "extremal spaces" to provide boundedness in the "intermediate spaces". However, while complex interpolation of L^p spaces requires the extremal spaces to also be L^p spaces, the Marcinkiewicz Interpolation Theorem only needs them to be weak-L^p spaces. It actually requires a bit less, the following definitions will make the exposition easier:

Definition 4.2. Let T be an operator defined on a subset of the set of measurable functions of a measure space (X, μ) and that takes values in the measurable functions of another measure space (Y, ν).

- We will say that T is of strong type (p, q) if T extends to a bounded operator from $L^p(X, \mu)$ to $L^q(Y, \nu)$.
- We will say that T is of weak type (p, q) if T extends to a bounded operator from $L^p(X, \mu)$ to $L^{q,\infty}(Y, \nu)$.

In what follows we will always assume T to be an operator defined on a set D closed under addition, multiplication by scalars and containing $\Sigma_c(X, \mu)$: the set of finite linear combinations of characteristic functions of sets of finite measure of (X, μ).

We will call T sublinear if

$$\left|T(f+g)\right| \leq |T(f)| + |T(g)|, \quad |T(\lambda f)| = |\lambda||T(f)|$$

for all f, g in the domain of T and $\lambda \in \mathbb{C}$.

We say that a sublinear operator T is of restricted weak type (p, q) if

$$\|Tf\|_{L^{q,\infty}} \lesssim HW^{1/p} \tag{14}$$

for all sub-step functions $f \in D$ of height H and width W.

The next result shows that, usually, an operator which is of restricted weak type on characteristic functions is also of restricted weak type on Σ_c. More precisely:

Proposition 4.3. *Let $0 < p \leq \infty$, $0 < q \leq \infty$, $A > 0$ and T be a sublinear operator defined on Σ_c. Then the following statements are equivalent:*

(1) *T satisfies*

$$\|Tf\|_{L^{q,\infty}} \lesssim AHW^{1/p}$$

for all sub-step functions f of height H and width W in Σ_c.

(2) *For every set $F \subseteq Y$ of finite measure there exists a subset $F' \subseteq F$ with $\nu(F') \geq \frac{1}{2}\nu(F)$ and such that for all $E \subseteq X$ of finite measure*

$$\int_Y |T\mathbb{1}_E|\mathbb{1}_{F'}\, d\nu \lesssim A\mu(E)^{1/p}\nu(F')^{1/q'}.$$

Proof. That (1) $\implies$ (2) is an easy consequence of Theorem 4.1 and Hölder's inequality for Lorentz spaces, so we will prove the reverse implication.

First assume that f takes the form

$$f_N = \sum_{j=1}^{N} 2^{-j}\mathbb{1}_{E_j}$$

where $\mu(E_j) \leq W$. In this case $\|Tf_N\|_{L^{q,\infty}} \lesssim AW^{1/p}$ (the constant being independent of N). To show this first observe that using sublinearity and Lemma 5.1 it suffices to show

$$\|T\mathbb{1}_{E_j}\|_{L^{q,\infty}} \lesssim AW^{1/p}. \tag{15}$$

Now let $F \subseteq Y$ be a set of finite measure. By hypothesis there exists a subset $F' \subseteq F$ which satisfies $\nu(F') \geq \frac{1}{2}\nu(F)$ and

$$\begin{aligned}\int_Y \mathbb{1}_{F'}T(\mathbb{1}_{E_j})\, d\nu &\lesssim A\mu(E_j)^{1/p}\nu(F')^{1/q'} \\ &\leq \nu(F')^{1/q'}AW^{1/p}.\end{aligned}$$

So by Theorem 4.1 we obtain (15).

Now let $0 \leq f \leq 1$ be an arbitrary function in Σ_c of width W:

$$f = \sum_{j=1}^{M} a_j\mathbb{1}_{E_j},$$

the E_j's disjointly supported. If $d_j(x)$ denotes the j-th digit in the binary expansion of $f(x)$ and we define

$$f_N = \sum_{j=1}^{N} 2^{-j}d_j = \sum_{j=1}^{N} 2^{-j}\mathbb{1}_{F_j},$$

then we easily see that

$$f - f_N = \sum_{j=1}^{M} b_{j,N}\mathbb{1}_{E_j}$$

with $0 \le b_{j,N} \le 2^{-N}$. So:

$$\begin{aligned}\|Tf\|_{L^{q,\infty}} &\lesssim \|Tf_N\|_{L^{q,\infty}} + \|T(f-f_N)\|_{L^{q,\infty}}\\ &\lesssim AW^{1/p} + \left\|\sum_{j=1}^{M} b_{j,N}|T\mathbb{1}_{E_j}|\right\|_{L^{q,\infty}}\\ &\lesssim AW^{1/p} + 2^{-N}\left\|\sum_{j=1}^{M} |T\mathbb{1}_{E_j}|\right\|_{L^{q,\infty}},\end{aligned}$$

since the second term can be made arbitrarily small we conclude that

$$\|Tf\|_{L^{q,\infty}} \lesssim AW^{1/p}.$$

By homogeneity and sublinearity we obtain the same bound (with a possibly larger constant) for arbitrary $f \in \Sigma_c$. □

To prove the Marcinkiewicz interpolation theorem we first need a weaker result:

Proposition 4.4. *Let $0 < p_0, p_1, q_0, q_1 \le \infty$, $A_0, A_1 > 0$ and suppose that T is a sublinear operator defined on Σ_c which is of restricted weak type (p_i, q_i) with constants A_i for $i = 0, 1$. Then it is of restricted weak type (p_θ, q_θ), i.e.:*

$$\|Tf\|_{L^{q_\theta,\infty}} \lesssim_{p_0,q_0,p_1,q_1} A_\theta H W^{1/p_\theta} \quad \forall f \in \Sigma_c, \tag{16}$$

where

$$\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q_\theta} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$$

and

$$A_\theta = A_0^{1-\theta} A_1^{\theta}$$

Proof. Theorem 4.3 reduces matters to showing that for every set $F \subseteq Y$ of finite measure there exists a subset $F' \subseteq F$ with $\nu(F') \ge \frac{1}{2}\nu(F)$ and such that for all sets $E \subseteq X$ of finite measure:

$$\int_Y \mathbb{1}_{F'} T(\mathbb{1}_E)\, d\nu \lesssim_{p_0,q_0,p_1,q_1} A_\theta \mu(E)^{1/p_\theta} \nu(F')^{1/q'_\theta},$$

but, using Theorem 4.3 and the hypotheses, we have that there exists a subset F' of F with $\nu(F') \ge \frac{1}{2}\nu(F)$ and that for all sets E of finite measure

$$\int_Y \mathbb{1}_{F'} T(\mathbb{1}_E)\, d\nu \lesssim_{p_0,q_0,p_1,q_1} A_\theta \mu(E)^{1/p_i} \nu(F')^{1/q'_i}$$

for $i = 0, 1$. The result now follows from the trivial scalar interpolation inequality:

$$X \le Y \quad \wedge \quad X \le Z \implies X \le Y^{1-\theta} Z^{\theta}$$

for $X, Y, Z \ge 0$ and $0 \le \theta \le 1$, and an application of Theorem 4.3 once more. □

We need one last lemma:

Lemma 4.5. *Let*

$$\Lambda_\lambda(x,y) = (1-\lambda)x + \lambda y,$$

then

$$\Lambda_\alpha(\Lambda_\beta(x,y), \Lambda_\gamma(x,y)) = \Lambda_{\Lambda_\alpha(\beta,\gamma)}(x,y).$$

We are now ready to state and prove the main theorem of this section:

Theorem 4.6. *Let T be a sublinear operator defined on Σ^+. Suppose that $0 < p_i, q_i \leq \infty$ and $q_0 \neq q_1$. If T satisfies*

$$\|T\mathbb{1}_E\|_{L^{q_i,\infty}} \lesssim A_i \|\mathbb{1}_E\|_{L^{p_i,1}} \quad i = 0, 1,$$

for all E of finite measure, then for all $1 \leq r \leq \infty$ and $0 < \theta < 1$ such that $q_\theta > 1$ we have

$$\|Tf\|_{L^{q_\theta,r}} \lesssim_{p_0,p_1,q_0,q_1,r,\theta} A_\theta \|f\|_{L^{p_\theta,r}}$$

for all f in Σ_c.

Proof. Using Theorem 4.3 we can assume that T is of restricted weak type (p_i, q_i) with constants A_i.

We will suppose first that $q_i \geq 1$. Let $f \in \Sigma^+$, which by homogeneity we may assume to be normalized so that $\|f\|_{L^{p_\theta,r}} = 1$. By duality (Theorem 3.3), it will be enough to verify the estimate

$$\int_Y |Tf| g \, d\nu \lesssim_{p_0,q_0,p_1,q_1,r,\theta} A_\theta$$

for all $g \in \Sigma^+$ with $\|g\|_{L^{q'_\theta,r'}} = 1$. In what follows all constants will be assumed to depend on p_0, q_0, p_1, q_1, r and θ and will be omitted.

By hypothesis and Hölder's inequality we have

$$\int_Y |Tu| v \, d\nu \lesssim A_i H H' W^{1/p_i} W'^{1/q'_i} \leq A_\theta H H' \min_{i=0,1} \left(W^{1/p_i} W'^{1/q'_i} \right) \tag{17}$$

for all sub-step functions $u \in \Sigma^+(X, \mu)$ of height H and width W and for all sub-step functions $v \in \Sigma^+(Y, \nu)$ of height H' and width W'.

By Theorem 2.2 we can decompose f and g as

$$f = \sum_m f_m \quad \text{and} \quad g = \sum_n g_n$$

where each f_m is a sub-step function of height H_m and width 2^m and where each g_n is a sub-step function of height H'_n and width 2^n. These decompositions also satisfy

$$\left\| H_m 2^{m/p_\theta} \right\|_{\ell^r} \sim \left\| H'_n 2^{n/q'_\theta} \right\|_{\ell^{r'}} \sim 1.$$

Recall that, since f and g are simple, only a finite number of f_m's and g_n's are non-zero.

Using (17) and sublinearity we have

$$\int_Y |Tf| g \, d\nu \lesssim A_\theta \sum_{m,n} H_m H'_n \min_{i=0,1} \left(2^{m/p_i} 2^{n/q'_i} \right).$$

Call $a_m = H_m 2^{m/p_\theta}$ and $b_n = H'_n 2^{n/q'_\theta}$, we then have to obtain the estimate

$$\sum_{m,n} a_m b_n \min_{i=0,1} \left(2^{m(1/p_i - 1/p_\theta)} 2^{n(1/q_\theta - 1/q_i)} \right) \lesssim 1, \tag{18}$$

where $\|a_m\|_{\ell^r} = \|b_n\|_{\ell^{r'}} = 1$.

After reorganizing terms we can write the third factor as

$$\min \left(m \left(\frac{1}{p_0} - \frac{1}{p_\theta} \right) + n \left(\frac{1}{q_\theta} - \frac{1}{q_0} \right), -m \left(\frac{1}{p_\theta} - \frac{1}{p_1} \right) - n \left(\frac{1}{q_1} - \frac{1}{q_\theta} \right) \right),$$

which can be further simplified to

$$\min \left(\alpha\theta \left(m - \frac{\beta}{\alpha} n \right), -\alpha(1-\theta) \left(m - \frac{\beta}{\alpha} n \right) \right),$$

where $\alpha = \frac{1}{p_0} - \frac{1}{p_1}$ and $\beta = \frac{1}{q_0} - \frac{1}{q_1}$, which is easily seen to be true. Let

$$\varphi(m - \gamma n) = \min \left(\alpha\theta \left(m - \gamma n \right), -\alpha(1-\theta) \left(m - \gamma n \right) \right)$$

where $\gamma = \frac{\beta}{\alpha}$, then we have to show that

$$\sum_{m,n} a_m b_n \varphi(m - \gamma n) \lesssim 1.$$

We can change variables and let $m \mapsto m + \lfloor \gamma n \rfloor$ and reduce to showing

$$\sum_{m,n} a_{m+\lfloor \gamma n \rfloor} b_n \varphi(m + \lfloor \gamma n \rfloor - \gamma n) \lesssim 1.$$

Observe that $|\lfloor \gamma n \rfloor - \gamma n| \leq 1$, so we easily see that

$$\varphi(m + \lfloor \gamma n \rfloor - \gamma n) \sim \varphi(m),$$

thus we end up with

$$\sum_m \varphi(m) \sum_n a_{m+\lfloor \gamma n \rfloor} b_n \lesssim \Big\| \sum_n a_{m+\lfloor \gamma n \rfloor} b_n \Big\|_{\ell^\infty_m}$$

since φ is in ℓ^1. Applying Hölder's inequality with exponents r and r' we reduce to showing that

$$\|a_{m+\lfloor \gamma n \rfloor}\|_{\ell^r_n} \lesssim 1$$

uniformly in m, but this is just the observation that the sum of $a^r_{m+\lfloor \gamma n \rfloor}$ is bounded by a constant depending on γ times the sum of a^r_{m+n}, which is of course bounded by a constant independent of m.

Finally, we show how to drop the assumption of $q_i > 1$. Observe that we may interpolate using Theorem 4.4 to obtain restricted weak type (p_θ, q_θ) boundedness of T for $0 \leq \theta \leq 1$. Since we are imposing $q_\theta > 1$, we may assume that $q_i > 1$ for at least one $i \in \{0, 1\}$ (otherwise the theorem is void), thus, there must be a $\tau \in (0, 1)$ such that $1 < q_\tau < q_\theta$ and such that T is of restricted weak type (p_τ, q_τ). We can now interpolate using what we have just proved from (p_τ, q_τ) to the (p_i, q_i) which has $q_i > 1$ and use Lemma 4.5 to ensure that (p_θ, q_θ) is in between (p_τ, q_τ) and (p_i, q_i).

□

Corollary 4.7 (Marcinkiewicz)**.** *Let T be a sublinear operator that is of weak type (p_i, q_i) with constants $A_i > 0$ and such that $1 \leq p_i \leq p_i \leq \infty$ for $i \in \{0, 1\}$, suppose also that $q_0 \neq q_1$. Then, for $0 < \theta < 1$, T is of strong type (p_θ, q_θ) with constant $\lesssim_{p_0,q_0,p_1,q_1,r,\theta} A_\theta = A_0^{1-\theta} A_1^\theta$ where*

$$\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \quad \textit{and} \quad \frac{1}{q_\theta} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

Proof. One just needs to recall that the $L^{p,q}$ are nested and that $L^{p,p}$ coincides with L^p, we leave the easy details to the reader. □

5. Appendix

Lemma 5.1. *Let X be a vector space endowed with a quasi-norm $\|\cdot\|$, that is, there exists a constant $C_1 > 0$ such that*

$$\|f + g\| \leq C_1(\|f\| + \|g\|). \tag{19}$$

Then, for any sequence $\{f_n\}_{n\in\mathbb{N}}$ of elements of X verifying

$$\|f_j\| \lesssim A C_2^{-j} \tag{20}$$

for some $A > 0$ and $C_2 > 1$, we have

$$\Big\| \sum_{j=1}^N f_j \Big\| \leq A C_4 \tag{21}$$

*where C_3 does **not** depend on N or A.*

Proof. Choose $\kappa \in \mathbb{N}$ so large that $C_1 C_2^{-\kappa} < 1$ and write N as

$$N = \kappa d + r$$

where $d \in \mathbb{N}$ and $0 \leq r < \kappa$. Observe that κ does not depend on N.

We can now split the sum as follows

$$\sum_{j=1}^{N} f_j = \sum_{j=1}^{\kappa d} f_j + \sum_{j=1}^{r} f_{kd+j}.$$

The norm of the second sum is trivially bounded by A times some constant depending on C_1 and C_2 (since $r < \kappa$,) so we can center our efforts on the first sum. To this end we factorize the sum as

$$\sum_{j=1}^{\kappa d} f_j = \sum_{m=0}^{d-1} \sum_{j=1}^{\kappa} f_{\kappa m+j}.$$

Now observe that, by induction, we have the following estimate:

$$\left\| \sum_{j=1}^{n} g_j \right\| \leq \sum_{j=1}^{n} C_1^j \|g_j\|$$

for any sequence $\{g_j\}$ of elements of X. Using this we can estimate the first sum as follows:

$$\begin{aligned}
\left\| \sum_{j=1}^{\kappa d} f_j + \sum_{j=1}^{r} f_{kd+j} \right\| &\leq \sum_{m=0}^{d-1} C_1^{m+1} \left\| \sum_{j=1}^{\kappa} f_{\kappa m+j} \right\| \\
&\leq A \sum_{m=0}^{d-1} C_1^{m+1} \sum_{j=1}^{\kappa} C_1^j C_2^{-(\kappa m+j)} \\
&= A \sum_{m=0}^{d-1} \left(C_1 C_2^{-\kappa}\right)^m \sum_{j=1}^{\kappa} C_1^{j+1} C_2^{-j} \\
&\leq A \sum_{m=0}^{\infty} \left(C_1 C_2^{-\kappa}\right)^m \cdot \sum_{j=1}^{\kappa} C_1^{j+1} C_2^{-j} \\
&\lesssim_{C_1,C_2} A \sum_{j=1}^{\kappa} C_1^{j+1} C_2^{-j} \\
&\lesssim_{C_1,C_2} A.
\end{aligned}$$

□

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