1. Introduction

If an operator $T$ is bounded on two Lebesgue spaces, the theory of complex interpolation allows us to deduce the boundedness of $T$ “between” the two spaces. Real interpolation (of $L^p$ spaces) provides, in a way, a refinement of complex interpolation, this will be shown as we progress. We will follow Tao’s notes [6]: in particular, we claim no originality over any argument here.

Consider the (centered) Hardy-Littlewood maximal operator:

\[(1) \quad Mf(x) = \sup_{r>0} \int_{B_r(x)} |f(y)| \, dy, \]

defined for $f \in L^1_{\text{loc}}$. We would like to obtain $L^p$ boundedness of this operator for $1 \leq p \leq \infty$, but alas, this is not the case: although it is trivially bounded from $L^\infty$ onto itself, it is not bounded on $L^1$:

**Proposition 1.1.** If $f \in L^1(\mathbb{R}^d)$ and $f$ is not identically 0 then

\[Mf(x) \geq \frac{C_d}{|x|^d}\]

for some constant $C_d$ dependent on $f$ and the dimension, in particular $f \notin L^1(\mathbb{R}^d)$.

**Proof.** Since $f$ is not identically 0, there exists an $R > 0$ such that

\[\int_{B_R(0)} |f| \geq \delta > 0.\]

Now, if $|x| > R$, $B_R(0) \subseteq B(x, 2|x|)$, so

\[Mf(x) \geq \frac{1}{|B(x, 2|x|)|} \int_{B_R(0)} |f| \geq C_d \frac{\delta}{|x|^d}.\]

Thus $M$ is indeed unbounded as an operator from $L^1$ onto itself. However, We do have the following weaker form of $L^1$ boundedness; proved by Hardy-Littlewood in [1] for $d = 1$ and by Wiener in [7] for $d > 1$:

**Theorem 1.2.** There exists a constant $C > 0$ such that for all $f \in L^1$

\[m(\{x \in \mathbb{R}^d : Mf(x) > \lambda\}) \leq C \frac{\|f\|_{L^1}}{\lambda},\]

where $m$ is the Lebesgue measure on $\mathbb{R}^d$. 

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Observe that, by Chebychev’s inequality, we would have had this bound if $M$ had been bounded on $L^1$, so it is indeed a weaker statement.

This leads us to study the space of functions satisfying Chebychev’s inequality:

$$\mu \left( \{ x \in X : | f(x) | > \lambda \} \right) \leq \frac{\| f \|_{L^p(X,\mu)}^p}{\lambda^p}.$$  \hfill (2)

More precisely, define the weak-$L^p$ space $L^{p,\infty}(X,\mu)$ to be the set of all measurable functions $f$ for which its $L^{p,\infty}$ quasi-norm

$$\| f \|_{L^{p,\infty}} = \sup_{\lambda > 0} \lambda \mu \left( \{ x : | f(x) | > \lambda \} \right)^{1/p}$$  \hfill (3)

is finite. To abbreviate we will use the notation

$$d_f(\lambda) = \mu \left( \{ x \in X : | f(x) | > \lambda \} \right)$$

when there is no confusion as to what measure we are using. Also, we define by convention $L^{\infty,\infty}$ to be $L^\infty$.

The following lemma is a simple application of the monotone convergence lemma:

**Lemma 1.3.** Let $\{ f_n \}_{n \in \mathbb{N}}$ be a sequence of measurable functions such that $| f_n(x) | \leq | f_{n+1}(x) |$ $\mu$-almost everywhere and for all $n \in \mathbb{N}$. If we have $| f_n | \to f$ $\mu$-almost everywhere for some function $f$ then

$$d_{f_n}(\lambda) \nearrow d_f(\lambda)$$

when $n \to \infty$ for all $\lambda > 0$.

As we have seen, Chebychev’s inequality gives us the inclusion $L^p \hookrightarrow L^{p,\infty}$ for $p > 0$. This inclusion is proper as the following example shows:

**Example 1.4.** If $f(x) = |x|^{-\frac{d}{2}}$ then

$$\int_{\mathbb{R}^d} |f(x)|^p \, dx = \int_{\mathbb{R}^d} |x|^{-d} \, dx = \infty,$$

however

$$\sup_{\lambda > 0} \lambda d_f(\lambda)^{1/p} = \nu_d^{1/p} < \infty,$$

where $\nu_d$ is the measure of the unit ball in $\mathbb{R}^d$.

As we have noted, $\| \cdot \|_{L^{p,\infty}}$ is not a priori a norm, but only a quasi-norm:

**Proposition 1.5.** We can easily see that

$$d_{f+g}(\lambda) \leq d_f(\lambda/2) + d_g(\lambda/2),$$

which is just the fact that when two terms sum more than $\lambda$, at least one of them has to be greater than $\lambda/2$. Using this fact we have:

$$\| f + g \|_{L^{p,\infty}} = \sup_{\lambda > 0} \lambda d_{f+g}(\lambda)^{1/p} \leq \sup_{\lambda > 0} \lambda \left( d_f(\lambda/2) + d_g(\lambda/2) \right)^{1/p} \leq 2C_p \left( \| f \|_{L^{p,\infty}} + \| g \|_{L^{p,\infty}} \right).$$

We will generalize the weak-$L^p$ spaces a bit further, bit first we need the following well-known identity:
**Proposition 1.6.** Let $0 < p < \infty$ and $f$ be a measurable function, then

$$
\|f\|_{L^p} = p^{1/p} \left( \int_{\mathbb{R}^+} \lambda^p d_f(\lambda) \frac{d\lambda}{\lambda} \right)^{1/p}.
$$

**Proof.**

$$
\|f\|_{L^p(X,\mu)}^p = \int_X |f(x)|^p d\mu(x)
= \int_X \int_{\mathbb{R}^+} |f(x)| p\lambda^p \frac{d\lambda}{\lambda} d\mu(x)
= p \int_{\mathbb{R}^+} \lambda^p \int_X \mathbb{1}_{\{|f(y)| > \lambda\}} d\mu(x) \frac{d\lambda}{\lambda}
= p \int_{\mathbb{R}^+} \lambda^p d_f(\lambda) \frac{d\lambda}{\lambda}.
$$

(Fubini)

So we have:

$$
\|f\|_{L^p,\infty} = \|d_f(\lambda)^{1/p}\|_{L^\infty(\mathbb{R}^+,\frac{d\lambda}{\lambda})}
$$

and

$$
\|f\|_{L^p} = p^{1/p} \|d_f(\lambda)^{1/p}\|_{L^p(\mathbb{R}^+,\frac{d\lambda}{\lambda})}.
$$

From this, we could tentatively define the

**Definition 1.7** (Lorentz quasi-norm). Let $(X,\mu)$ be a measure space and let $0 < p < \infty$ and $0 < q \leq \infty$, we define the Lorentz space quasi-norm $\|\cdot\|_{L^{p,q}(X,\mu)}$ as

$$
\|f\|_{L^{p,q}(X,\mu)} = p^{1/q} \|\lambda d_f(\lambda)^{1/p}\|_{L^q(\mathbb{R}^+,\frac{d\lambda}{\lambda})}.
$$

Observe that, with this definition, $L^{p,p}$ (isometrically) coincides with $L^p$.

We have the following useful lemma, which is a kind of Monotone Convergence Theorem.

**Lemma 1.8.** Let $\{f_n\}_{n \in \mathbb{N}}$ be a sequence of measurable functions such that $|f_n| \not\in \mathcal{L} |f| \mu$-almost everywhere, then

$$
\|f\|_{L^{p,q}} = \lim_{n \to \infty} \|f_n\|_{L^{p,q}}
$$

**Proof.** We can obviously assume $f$ to be positive so, by definition, we only have to verify that

$$
\lim_{n \geq 1} \|\lambda d_{f_n}(\lambda)^{1/p}\|_{L^q(\mathbb{R}^+,\frac{d\lambda}{\lambda})} = \|\lambda d_f(\lambda)^{1/p}\|_{L^q(\mathbb{R}^+,\frac{d\lambda}{\lambda})}.
$$

Note that, since $f_n$ is non-decreasing, $d_{f_n}(\cdot)$ is also increasing (in $n$), so in particular by Lemma 1.3

$$
d_f(\lambda) = \sup_{n \geq 1} d_{f_n}(\lambda)
$$

for all fixed $\lambda > 0$.

If $q = \infty$ then the problem is reduced to the easier one of showing that

$$
\sup_{n \geq 1} \|\lambda d_{f_n}(\lambda)^{1/p}\|_{L^\infty(\mathbb{R}^+)} = \sup_{n \geq 1} \|\lambda d_{f_n}(\lambda)^{1/p}\|_{L^\infty(\mathbb{R}^+)}.
$$

which is easily seen to hold true in the more general case

$$
\sup_{n \geq 1} \|g_n\|_{L^\infty} = \sup_{n \geq 1} \|g_n\|_{L^\infty}.
$$
Suppose now that $q < \infty$, then
\[
\sup_{n \geq 1} \|\lambda d f_n(\lambda)\|_{L^q(\mathbb{R}^+,\frac{d\lambda}{\lambda})}^{1/q} = \left( \sup_{n \geq 1} \int_0^\infty \lambda^q d f_n(\lambda)\frac{d\lambda}{\lambda} \right)^{1/q},
\]
and since, as we noted before, we have $\lambda^q d f_n(\lambda) \uparrow \lambda^q d f(\lambda)$ for all fixed $\lambda > 0$, we can apply the monotone convergence theorem with the measure $\frac{d\lambda}{\lambda}$ and the desired identity. □

The following Fatou-like lemma shows that the Lorentz space quasi-norm is lower semi-continuous.

**Lemma 1.9.** Let $\{f_n\}_{n \in \mathbb{N}}$ be any sequence of measurable functions, then
\[
\| \liminf_n f_n \|_{L^{p,q}} \leq \liminf_n \| f_n \|_{L^{p,q}}.
\]

**Proof.** Let $f = \liminf_n f_n$, then, by definition, we have to check that
\[
\|\lambda d f(\lambda)\|_{L^q(\mathbb{R}^+,\frac{d\lambda}{\lambda})} \leq \liminf_n \|\lambda d f_n(\lambda)\|_{L^q(\mathbb{R}^+,\frac{d\lambda}{\lambda})},
\]
which would follow from the lower-semicontinuity of the usual $L^q$ spaces provided that we have
\[
d f(\lambda) \leq \liminf_n d f_n(\lambda).
\]
This is equivalent to showing
\[
\mu(\{x : \liminf_n |f_n(x)| > \lambda\}) \leq \liminf_n \mu(\{x : |f_n(x)| > \lambda\}),
\]
which follows from Fatou’s lemma. □

Finally, we can prove that the $L^{p,q}$ spaces are quasi-Banach spaces; which follows the steps of the classical proof in the $L^p$ case.

**Theorem 1.10.** Let $(X,\mu)$ be a measure space and let $0 < p < \infty$, $0 < q \leq \infty$, then $L^{p,q}(X,\mu)$ is a quasi-Banach space, that is, it is complete and it satisfies the quasi-triangle inequality.

**Proof.** We have already proven that $L^{p,\infty}$ satisfies the quasi-triangle inequality in Proposition 1.5, the proof for $L^{p,q}$ when $q \neq \infty$ is almost identical so we will omit it.

We are left to prove that it is complete. To this end first note that the $L^{p,q}$ norm always controls the $L^{p,\infty}$ norm:
\[
\|\lambda d f(\lambda)\|_{L^q(\mathbb{R}^+,\frac{d\lambda}{\lambda})} = \left( \int_0^\infty \lambda^q d f(\lambda)\frac{d\lambda}{\lambda} \right)^{1/q} \geq \left( \int_0^s \lambda^q d f(\lambda)\frac{d\lambda}{\lambda} \right)^{1/q} \geq d f(s)^{1/p} \left( \int_0^s \lambda^q \frac{d\lambda}{\lambda} \right)^{1/q} = s d f(s)^{1/p} q^{-1/q},
\]
for any $s > 0$. Thus, taking the supremum on $s$, we obtain:
\[
\|f\|_{L^{p,\infty}} \lesssim_{p,q} \|f\|_{L^{p,q}}.
\]
In particular we have the following generalization of Chebychev’s inequality:
\[
d f(\lambda) \lesssim_{p,q} \|f\|_{L^{p,q}}^{1/q}.
\]

Now let $\{f_n\}$ be a Cauchy sequence in the $L^{p,q}$ quasi-norm. Estimate (4) allows us to deduce that $\{f_n\}$ is Cauchy in measure, and hence that there exists a $\mu$-almost everywhere convergent
subsequence \{f_{\varphi(k)}\}_k. Call \( f \) the \( \mu \)-almost everywhere defined limit of \( f_{\varphi(k)} \) when \( k \to \infty \), we will show that \( f_n \to f \) in the \( L^{p,q} \) quasi-norm.

Let \( C \) be the constant in the quasi-triangle inequality for the \( \| \cdot \|_{L^{p,q}} \) quasi-norm and refine the subsequence \{f_{\varphi(k)}\} so that we have
\[
\|f_{\varphi(k)} - f_{\varphi(k-1)}\|_{L^{p,q}} \leq (2C)^{-k}
\]
using that \( f_n \) is an \( L^{p,q} \)-Cauchy sequence and defining \( f_{\varphi(0)} = 0 \). Observe that we have the following telescoping sum identity:
\[
f_{\varphi(n)} = \sum_{k=1}^{n} f_{\varphi(k)} - f_{\varphi(k-1)}.
\]
(5)

Recall that we want to estimate \( \|f - f_n\|_{L^{p,q}} \) and that, using the quasi-triangle inequality, we have
\[
\|f - f_n\|_{L^{p,q}} \leq C \left( \|f - f_{\varphi(N)}\|_{L^{p,q}} + \|f_{\varphi(N)} - f_n\|_{L^{p,q}} \right).
\]
The second term can be made arbitrarily small using the fact that \( f_n \) is an \( L^{p,q} \)-Cauchy sequence, so we choose \( n \) and \( N \) so large that
\[
\|f_{\varphi(N)} - f_n\|_{L^{p,q}} < \frac{\epsilon}{2C}.
\]
For the first term we can use (5) to obtain:
\[
\|f - f_{\varphi(N)}\|_{L^{p,q}} = \left\| \sum_{k=1}^{N} (f_{\varphi(k)} - f_{\varphi(k-1)}) \right\|_{L^{p,q}}
\]
\[
= \left\| \lim_{M \to \infty} \sum_{k=N+1}^{M} (f_{\varphi(k)} - f_{\varphi(k-1)}) \right\|_{L^{p,q}}
\]
\[
\leq \left\| \lim_{M \to \infty} \sum_{k=N+1}^{M} |f_{\varphi(k)} - f_{\varphi(k-1)}| \right\|_{L^{p,q}}
\]
(6)

where in the last line we have used Proposition 1.9. By induction, we can iteratively use the quasi-triangle inequality to exchange \( \sum \) and \( \| \cdot \|_{L^{p,q}} \) and bound the last term in (6) by
\[
\lim_{M \to \infty} \sum_{k=N+1}^{M} C^{k-N} \|f_{\varphi(k)} - f_{\varphi(k-1)}\|_{L^{p,q}} \leq \lim_{M \to \infty} \sum_{k=N+1}^{M} C^{k-N} (2C)^{-k}
\]
\[
= C^{-N} \sum_{k=N+1}^{\infty} 2^{-k}
\]
\[
\leq (2C)^{-N}.
\]
Choosing \( N \) so large that \( \frac{1}{(2C)^{-N}} < \frac{\epsilon}{2C} \) completes the proof. \( \square \)

2. The Dyadic Decomposition

It turns out that we will be able to decompose functions in \( L^{p,q} \) into simpler functions:

- Sub-step functions: A sub-step function of height \( H \) and width \( W \) is any measurable function \( f \) supported on a set of measure \( W \) and which satisfies \( |f| \leq H \).
• Quasi-step function: A quasi-step function of height $H$ and width $W$ is any measurable function $f$ supported on a set of measure $W$ and which satisfies $|f| \sim H$ in its support.

**Example 2.1.** Let $f$ be a quasi-step function of height $H$ and width $W$, then

$$\|f\|_{L^{p,q}} \sim HW^{1/p} \left(\frac{p}{q}\right)^{1/q}.$$  

We will give two different decompositions: a “vertically-dyadic” decomposition; where we decompose the range of our function into dyadic pieces, and a “horizontally-dyadic” one; where we dyadically decompose its support.

Let $\{\gamma_m\}_{m \in \mathbb{Z}}$ be a sequence in $\mathbb{R}^+$, we say that it is of lower exponential growth if we have

$$C_1 \gamma_m \leq \gamma_{m+1}$$

for some constant $C_1 > 1$. We will say that it is of exponential growth if we additionally have

$$\gamma_{m+1} \leq C_2 \gamma_m$$

for some $C_2 > 1$.

Note that the condition of being of lower exponential growth can easily be tested by the equivalent condition of

$$\inf_{m \in \mathbb{Z}} \frac{\gamma_{m+1}}{\gamma_m} > 1,$$

while that of being of exponential growth is in turn equivalent to being of lower exponential growth and

$$\sup_{m \in \mathbb{Z}} \frac{\gamma_{m+1}}{\gamma_m} < \infty.$$  

We can now state the main theorem of this section:

**Theorem 2.2.** Let $f \geq 0$ be a measurable function, $\{\gamma_m\}_{m \in \mathbb{Z}}$ be a sequence of exponential growth and $0 < p < \infty$, $1 \leq q \leq \infty$. Then the following statements are equivalent:

(a) There exists a decomposition $f = \sum_{m \in \mathbb{Z}} f_m$, where each $f_m$ is a quasi-step function of height $\gamma_m$ and width $W_m$, the $f_m$’s have pairwise disjoint supports and we have

$$(7) \quad \|\gamma_m W_m^{1/p}\|_{L^q} \lesssim p,q 1.$$  

(b) We have the estimate

$$\|f\|_{L^{p,q}} \lesssim p,q 1.$$  

(c) There exists a decomposition $f = \sum_{m \in \mathbb{Z}} f_m$, where each $f_m$ is a sub-step function of width $\gamma_m$ and height $H_m$, the $f_m$’s have pairwise disjoint supports, $H_m$ is non-increasing, $H_{m+1} \leq |f_m| \leq H_m$ in the support of $f_m$ and we have

$$(8) \quad \|H_m \gamma_m^{1/p}\|_{L^q} \lesssim p,q 1.$$  

**Proof.** Observe that, since $\{\gamma_m\}_{m \in \mathbb{Z}}$ is of lower exponential growth, we have

$$(9) \quad \gamma_{m-k} \leq C_1^{-k} \gamma_m \quad \text{and} \quad \gamma_{m+k} \geq C_1^{k} \gamma_m$$

for all $m \in \mathbb{Z}$ and $k \geq 0$. This implies in particular that $\gamma_m \to 0$ when $m \to -\infty$ and $\gamma_m \to \infty$ when $m \to \infty$.

Let us first prove that (a) $\Rightarrow$ (b): By monotonicity we can assume that

$$f = \sum_{m \in \mathbb{Z}} \gamma_m 1_{E_m}.$$
where $E_m$ is the support of $f_m$. Since the sequence $\gamma_m$ is (strictly) increasing, we have

$$d_f(\gamma_m) = \sum_{k=1}^{\infty} \mu(E_{m+k}).$$

Let us first suppose that $q = \infty$, we have to prove that

$$\sup_{\lambda > 0} \lambda d_f(\lambda)^{1/p} \lesssim_p 1.$$

To this end observe that

$$\sup_{\lambda > 0} \lambda d_f(\lambda)^{1/p} = \sup_{m \in \mathbb{Z}} \sup_{\lambda \in (\gamma_m, \gamma_{m+1}]} \lambda d_f(\lambda)^{1/p}$$

$$\leq \sup_{m \in \mathbb{Z}} \gamma_{m+1} d_f(\gamma_m)^{1/p}$$

$$= \sup_{m \in \mathbb{Z}} \gamma_{m+1} \left( \sum_{k=1}^{\infty} W_{m+k} \right)^{1/p}$$

$$\leq \left( \sum_{k=1}^{\infty} \sup_{m \in \mathbb{Z}} \gamma_{m+1-k} W_m \right)^{1/p}$$

$$= \left( \sum_{k=1}^{\infty} \sup_{m \in \mathbb{Z}} \gamma_{m+1-k} W_m \right)^{1/p}$$

$$\leq \sup_{m \in \mathbb{Z}} \sup_{\lambda \in (\gamma_m, \gamma_{m+1]}} \lambda^q d_f(\lambda)^{q/p} d\lambda$$

$$\lesssim_p \sup_{m \in \mathbb{Z}} \sup_{\lambda \in (\gamma_m, \gamma_{m+1]}} \lambda^q d_f(\lambda)^{q/p} d\lambda.$$

Now suppose $q < \infty$, then we can do something similar:

$$\|f\|_{L^{p,q}} \sim_p \| \lambda d_f(\lambda)^{1/p} \|^q_{L^q(\mathbb{R}^+, \frac{d\lambda}{\lambda})}$$

$$= \sum_{m \in \mathbb{Z}} \int_{\gamma_m}^{\gamma_{m+1}} \lambda^q d_f(\lambda)^{q/p} \frac{d\lambda}{\lambda}$$

$$\leq \sum_{m \in \mathbb{Z}} d_f(\gamma_m)^{q/p} \int_{\gamma_m}^{\gamma_{m+1}} \lambda^q \frac{d\lambda}{\lambda}$$

$$\lesssim_q \sum_{m \in \mathbb{Z}} d_f(\gamma_m)^{q/p} = \left( \sum_{k=1}^{\infty} \sup_{m \in \mathbb{Z}} \gamma_{m+1-k} W_m \right)^{1/p}$$

$$= \left( \sum_{k=1}^{\infty} \gamma_{m+1-k} W_m \right)^{1/p}.$$
From (9) and (7), we have:

\[
\|\gamma_{m+1}^p W_{m+k}\|_{\ell^{q/p}_m} = \|\gamma_{m-(k-1)}^p W_m\|_{\ell^{q/p}_m} \\
\leq C_1^{-p(k-1)}\|\gamma_m^p W_m\|_{\ell^{q/p}_m} \\
\lesssim C_1^{-pk}\|\gamma_m^p W_m\|_{C^{q/p}_m} \\
= C_1^{-pk}\|\gamma_m^p W_{m+1/p}\|_{C^{q/p}_m} \\
\lesssim p C_1^{-pk}.
\]

Hence, using Lemma 5.1, we obtain:

\[
\left\| \sum_{k=1}^{\infty} \gamma_{m+1}^p W_{m+k} \right\|_{\ell^{q/p}_m} \lesssim C_{1,p,q} 1.
\]

Let us now see that (b) \(\Rightarrow\) (c):

Define \(H_m = \inf\{\lambda > 0 : d_f(\lambda) \leq \gamma_m\}\),

Clearly \(H_m\) is non-increasing and, by definition, we have

\[\lambda < H_m \Rightarrow d_f(\lambda) > \gamma_m.\]

Observe also that, from the monotone convergence theorem, this infimum is achieved and so \(d_f(H_m) = \gamma_m\). Since \(\|f\|_{L^{p,q}}\) is finite, \(d_f(\lambda)\) has to be finite for \(\lambda > 0\) (otherwise \(d_f(s)\) would be infinite for \(0 < s < \lambda\)). Thus, if \(H_m \neq 0\), then the sets

\[E_m = \{x : f(x) > H_m\}\]

are of finite measure. Using the monotone convergence theorem again we conclude that the set

\[E = \{x : f(x) \geq \sup_m H_m\}\]

has measure 0. By the definition of \(H_m\)

\[d_f(\lambda) \leq \gamma_m \Rightarrow H_m \leq \lambda,\]

thus if \(d_f(\epsilon) \leq \gamma_m\) (which is true for \(m\) sufficiently large since \(\gamma_m \to \infty\) when \(m \to \infty\)), then \(H_m \leq \epsilon\), that is \(H_m \to 0\) when \(m \to \infty\).

These remarks show that

\[f = \sum_{m \in \mathbb{Z}} f_m,\]

where \(f_m = f\mathbb{1}_{H_m+1 < f \leq H_m}\).

If \(q < \infty\) then

\[
\|\lambda d_f(\lambda)^{1/p}\|^q_{L^q((H_m+k+1, H_{m+k+1}], \frac{dx}{x})} \geq \gamma_{m+k}^q \int_{H_{m+k+1}}^{H_{m+k+1}} \lambda^q d\lambda \\
\sim q \int_{H_{m+k}+1}^{H_{m+k+1}} H_{m+k}^q - H_{m+k+1}^q \\
\sim_{C_{1,p,q}} C_1^{-q/p} \gamma_m^q (H_{m+k}^q - H_{m+k+1}^q).
\]
Since $H_m \to 0$ when $m \to \infty$, we can sum telescopically as follows:

$$H_m^{1/p} = \left( \frac{H_m^q}{H_m^{q/p}} \right)^{1/q}$$

$$= \left( \sum_{k=0}^{\infty} (H_{m+k}^q - H_{m+k+1}^q)_{m+k}^{q/p} \right)^{1/q}$$

$$\lesssim_{C_1,q} \left( \sum_{k=0}^{\infty} C_1^{kq/p} \| \lambda d_f (\lambda) \|_{L^q((H_{m+k+1}, H_{m+k+k}), \frac{d\lambda}{\lambda})} \right)^{1/q},$$

so taking $\ell^q$ norms

$$\| H_m^{1/p} \|_{\ell^q} \lesssim_{C_1,p,q} \sum_{m \in \mathbb{Z}} \sum_{k=0}^{\infty} C_1^{-kq/p} \| \lambda d_f (\lambda) \|_{L^q((H_{m+k+1}, H_{m+k+k}), \frac{d\lambda}{\lambda})}$$

$$= \sum_{k=0}^{\infty} C_1^{-kq/p} \sum_{m \in \mathbb{Z}} \| \lambda d_f (\lambda) \|_{L^q((H_{m+k+1}, H_{m+k+k}), \frac{d\lambda}{\lambda})}$$

$$= \sum_{k=0}^{\infty} C_1^{-kq/p} \| \lambda d_f (\lambda) \|_{L^q(\mathbb{R}^+, \frac{d\lambda}{\lambda})}$$

$$\lesssim_{C_1,p,q} 1.$$

Lastly, let $q = \infty$. Then we immediately have

$$d_f (\lambda) \lesssim \lambda^{-p},$$

so if $\lambda > \gamma_m^{-1/p}$, then $d_f (\lambda) \lesssim \gamma_m^{-1/p}$. This implies

$$\| H_m \|_{\ell^\infty} \lesssim 1,$$

which is what we wanted.

We now prove the implications in the reverse order. Let’s start with (c) $\implies$ (b):

By hypothesis and the estimate (9) we have

$$d_f (H_m) \leq \sum_{k=1}^{\infty} \gamma_m - k \leq \sum_{k=1}^{\infty} C_1^{-k} \lesssim_{C_1} \gamma_m.$$

Observe also that it follows from (7) that $H_m \to 0$ when $m \to \infty$. Additionally:

$$\lambda < \sup_{m \in \mathbb{Z}} H_m \implies \lambda \in \bigcup_{m \in \mathbb{Z}} [H_{m+1}, H_m).$$

So, if $q < \infty$:

$$\| \lambda d_f (\lambda) \|_{L^q(\mathbb{R}^+, \frac{d\lambda}{\lambda})} = \sum_{m \in \mathbb{Z}} \int_{H_{m+1}}^{H_m} \lambda^q d_f (\lambda) \frac{d\lambda}{\lambda}$$

$$\lesssim_{C_1,q} \sum_{m \in \mathbb{Z}} \gamma_m^{q/p} (H_m^q - H_{m+1}^q)$$

$$\lesssim_{C_2,p} \sum_{m \in \mathbb{Z}} \gamma_m^{q/p} H_m$$

$$= \| H_m \|_{\ell^q}$$

$$\lesssim 1.$$
If $q = \infty$ then we can similarly estimate:

$$
\sup_{\lambda > 0} \lambda d_f(\lambda)^{1/p} = \sup_{m \in \mathbb{Z}} \sup_{\lambda \in (H_{m+1}, H_m)} \lambda d_f(\lambda)^{1/p}
\leq \sup_{m \in \mathbb{Z}} H_m d_f(H_{m+1})^{1/p}
\lesssim C_1 p \sup_{m \in \mathbb{Z}} H_m^{1/p} 
\lesssim C_2 p \sup_{m \in \mathbb{Z}} H_m^{1/p}
\lesssim 1.
$$

The final stage of the proof is showing that

$$(b) \implies (a).$$

Define

$$f_m = f_{\gamma_m < |f| \leq \gamma_{m+1}},$$

then clearly

$$W_m = d_f(\gamma_m) - d_f(\gamma_{m+1}) \leq d_f(\gamma_m).$$

If $q < \infty$ we proceed as follows:

$$1 \gtrsim \|\lambda d_f(\lambda)^{1/p}\|_{L^q(\mathbb{R}^+)}^q = \sum_{m \in \mathbb{Z}} \int_{\gamma_m}^{\gamma_{m+1}} \lambda^q d_f(\lambda)^{q/p} d\lambda
\geq \sum_{m \in \mathbb{Z}} d_f(\gamma_m)^{q/p} (\gamma_{m+1}^q - \gamma_m^q)
\gtrsim C_2 \sum_{m \in \mathbb{Z}} W_m^{q/p-q}
= \|\gamma_m W_m^{1/p}\|_{L^q}^q.$$

If $q = \infty$, then we have to show that

$$\sup_m \gamma_m W_m^{1/p} \lesssim 1,$$

but by hypothesis $d_f(\lambda)^{1/p} \lesssim \lambda^{-1}$, so

$$W_m^{1/p} \leq d_f(\gamma_m)^{1/p} \lesssim \gamma_m^{-1}$$

and the proof follows. $\square$

**Remark 2.3.** Observe that the proof of $(a) \implies (b)$ works verbatim when the sequence $\{\gamma_m\}_{m \in \mathbb{Z}}$ is only of lower exponential growth.

### 3. Quantitative Properties of $L^{p,q}$ Spaces

As a first application of the dyadic decomposition theorems of the last section we can prove the following generalization of Hölder’s inequality, originally proven by O’Neil in [4].

**Theorem 3.1.** Let $0 < p, p_1, p_2 < \infty$ and $0 < q, q_1, q_2 \leq \infty$ exponents satisfying

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \quad \text{and} \quad \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2},$$

then
then for every measurable function we have
\[ \|fg\|_{L^{p,q}} \lesssim_p \|f\|_{L^{p_1,q_1}}\|g\|_{L^{p_2,q_2}}. \]

**Proof.** First observe that we can assume \( \|f\|_{L^{p_1,q_1}} = \|g\|_{L^{p_2,q_2}} = 1 \) and that \( f, g \geq 0 \), so we only have to prove
\[ \|fg\|_{L^{p,q}} \lesssim_p 1. \]
We will omit the dependence on the exponents for the rest of the proof.

Decompose \( f \) and \( g \) using the horizontally dyadic decomposition with \( \gamma_m = 2^m \), i.e.:
\[ f = \sum_m f_m, \quad g = \sum_m g_m \]
where \( f_m \) and \( g_m \) are sub-step functions of height \( H_m \) and \( H'_m \) respectively and have supports of measure \( \leq 2^m \).

To estimate the \( fg \) we proceed as follows:
\[ fg \leq \sum_m \sum_n f_m g_n \]
\[ = \sum_k \sum_m f_m g_{m+k} \]
\[ = \sum_{k<0} \sum_m f_m g_{m+k} + \sum_{k\geq 0} \sum_m f_m g_{m+k}, \]

since the \( L^{p,q} \) quasi-norm satisfies the quasi-triangle inequality it suffices to estimate the \( k < 0 \) and \( k \geq 0 \) sums separately. We will only show
\[ \| \sum_{k \geq 0} \sum_m f_m g_{m+k} \|_{L^{p,q}} \lesssim_p 1 \]
since the other sum is estimated in the same way.

Observe that \( f_m g_{m+k} \) is a sub-step function of height \( H_m H'_{m+k} \) and width \( \leq \min(2^m, 2^{m+k}) \), so
\[ \sum_m f_m g_{m+k} \leq \sum_m H_m H'_{m+k}^\beta e_m, \]
where \( E_m \) is a set of measure \( 2^m \). We can now employ the implication \( (c) \implies (b) \) in Theorem 2.2 to obtain:
\[ \| \sum_m f_m g_{m+k} \|_{L^{p,q}} \lesssim_p \left\| H_m H'_{m+k}^{2m/p} \right\|_{\ell^q} \]
\[ = \left\| 2^{m/p} H_m 2^{m/p} H'_{m+k} \right\|_{\ell^q} \]
\[ \leq \left\| 2^{m/p} H_m \right\|_{\ell^{q_1}} \left\| 2^{m/p} H'_{m+k} \right\|_{\ell^{q_2}} \]
\[ \lesssim 2^{-k/p_2}, \]
where we have used Hölder’s inequality in the last line.

To get
\[ \left\| \sum_{k \geq 0} \sum_m f_m g_{m+k} \right\|_{L^{p,q}} \lesssim_p 1 \]
we just use Lemma 5.1. \( \square \)

The following dual characterization of the \( L^{p,\infty} \) space will be useful.

**Theorem 3.2.** Let \((X, \mu)\) be a \( \sigma \)-finite measure space, \( 1 < p < \infty \) and \( f \in L^{p,\infty} \). Then
(i) If \( f \in L^{p, \infty} \) then 
\[
\|f\|_{L^{p, \infty}} \sim_p \sup_{0 < \mu(E) < \infty} \frac{1}{\mu(E)^{1/p'}} \left| \int_X f \mathbb{1}_E \, d\mu \right|.
\] 
(ii) If \( f \mathbb{1}_E \) is integrable for all sets \( E \) of finite measure and 
\[
M_p = \sup_{0 < \mu(E) < \infty} \frac{1}{\mu(E)^{1/p'}} \left| \int_X f \mathbb{1}_E \, d\mu \right| < \infty,
\] 
then \( f \in L^{p, \infty} \) and \( \|f\|_{L^{p, \infty}} \lesssim_p M_p \).

Proof. Let’s prove (i) first. The \( \gtrsim_p \) part is a simple consequence of Hölder’s inequality for Lorentz spaces, so we will concentrate on showing the reverse inequality.

Let us first assume \( f \) to be non-negative. Let \( E_\lambda = \{ x \in X : f(x) > \lambda \} \), the assumption \( f \in L^{p, \infty} \) guarantees that \( \mu(E_\lambda) < \infty \). Then, if \( \mu(E_\lambda) \neq 0 \):
\[
\lambda \mu(E_\lambda)^{1/p} = \frac{1}{\mu(E_\lambda)^{1/p'}} \mu(E_\lambda) \leq \frac{1}{\mu(E_\lambda)^{1/p'}} \int_X f \mathbb{1}_{E_\lambda} \, d\mu \leq \sup_{0 < \mu(E) < \infty} \frac{1}{\mu(E)^{1/p'}} \left| \int_X f \mathbb{1}_E \, d\mu \right|,
\]
taking the supremum in \( \lambda > 0 \) yields (10).

Now suppose \( f \) is not necessarily non-negative. We can decompose \( f = u^+_0 - u^-_0 + i(u^+_1 - u^-_1) \) where \( u^+_i \geq 0 \) for \( i \in \{0, 1\} \), so by the quasi-triangle inequality:
\[
\|f\|_{L^{p, \infty}} \lesssim_p \|u^+_0\|_{L^{p, \infty}} + \|u^-_0\|_{L^{p, \infty}} + \|u^+_1\|_{L^{p, \infty}} + \|u^-_1\|_{L^{p, \infty}}.
\]

It thus suffices to show that
\[
\|u^+_i\|_{L^{p, \infty}} \leq \sup_{0 < \mu(E) < \infty} \frac{1}{\mu(E)^{1/p'}} \left| \int_X f \mathbb{1}_E \, d\mu \right|,
\]
but since \( u^+_i \) is non-negative, we can use what we already know to reduce the problem to showing that
\[
\sup_{0 < \mu(E) < \infty} \frac{1}{\mu(E)^{1/p'}} \left| \int_X u^+_i \mathbb{1}_E \, d\mu \right| \leq \sup_{0 < \mu(E) < \infty} \frac{1}{\mu(E)^{1/p'}} \left| \int_X f \mathbb{1}_E \, d\mu \right|.
\]

Finally, observe that \( u^+_i = e^{i\theta} f \mathbb{1}_F \) for some \( F \subseteq X \), so for any given set \( E \) with \( 0 < \mu(E) < \infty \) we have that
\[
\frac{1}{\mu(E)^{1/p'}} \left| \int_X u^+_i \mathbb{1}_E \, d\mu \right| = 0 \leq \sup_{0 < \mu(E') < \infty} \frac{1}{\mu(E')^{1/p'}} \left| \int_X e^{i\theta} f \mathbb{1}_{E \cap F} \, d\mu \right|
\]
if \( \mu(E \cap F) = 0 \) and
\[
\frac{1}{\mu(E)^{1/p'}} \left| \int_X u^+_i \mathbb{1}_E \, d\mu \right| = \frac{1}{\mu(E)^{1/p'}} \left| \int_X e^{i\theta} f \mathbb{1}_{E \cap F} \, d\mu \right| \leq \frac{1}{\mu(E \cap F)^{1/p'}} \left| \int_X f \mathbb{1}_{E \cap F} \, d\mu \right| \leq \sup_{0 < \mu(E') < \infty} \frac{1}{\mu(E')^{1/p'}} \left| \int_X f \mathbb{1}_E \, d\mu \right|
\]
if \( \mu(E \cap F) > 0 \). Taking the supremum over these sets yields the proof.

Let us now prove (ii). The idea is to approximate \( f \) by functions \( f_n \) which are a priori in \( L^{p, \infty} \), use (i) on these functions and then try to pass to the limit.

Indeed, since \( (X, \mu) \) is \( \sigma \)-finite we can find an increasing sequence of sets \( \{F_n\}_{n \in \mathbb{N}} \) of finite measure and such that
\[
X = \bigcup_n F_n.
\]
We now define \( f_n = f \mathbb{1}_{E_n} \) with \( E_n = E \cap \{ x : |f(x)| \leq n \} \). These functions are qualitatively in \( L^{p,\infty} \):

\[
\| f_n \|_{L^{p,\infty}} \lesssim n^{\mu(E_n)^{1/p}},
\]

and also |\( f_n \)\( f \)| \( f \), so we can use Proposition 1.9 to get:

(11)
\[
\| f \|_{L^{p,\infty}} \leq \liminf_{n \to \infty} \| f_n \|_{L^{p,\infty}}.
\]

We would have finished if the right hand side of (11) were finite, because then we would have \( f \in L^{p,\infty} \) a priori. But observe that we can use (i) to bound each of these functions:

\[
\| f_n \|_{L^{p,\infty}} \lesssim \sup_{0 < \mu(E) < \infty} \frac{1}{\mu(E)^{1/p'}} \left| \int_X f_n \mathbb{1}_E d\mu \right|.
\]

Now we use a procedure similar to that in the proof of (i):

\[
\sup_{0 < \mu(E) < \infty} \frac{1}{\mu(E)^{1/p'}} \left| \int_X f \mathbb{1}_{E_n} d\mu \right| \leq \sup_{0 < \mu(E) < \infty} \frac{1}{\mu(E \cap E_n)^{1/p'}} \left| \int_X f \mathbb{1}_{E \cap E_n} d\mu \right| = \sup_{\mu(E') > 0, E' \subseteq E_n} \frac{1}{\mu(E')^{1/p'}} \left| \int_X f \mathbb{1}_{E'} d\mu \right| \leq \sup_{0 < \mu(E') < \infty} \frac{1}{\mu(E')^{1/p'}} \left| \int_X f \mathbb{1}_{E'} d\mu \right| = M_p.
\]

We can also prove a duality theorem for \( L^{p,q} \) spaces:

**Theorem 3.3.** Let \( 1 < p < \infty \) and \( 1 \leq q \leq \infty \), then for all \( f \in \Sigma^+ \):

(12)
\[
\| f \|_{L^{p,q}} \sim_{p,q} \sup_{g \in \Sigma^+, \| g \|_{L^{p',q'}} \leq 1} \left\{ \int_X fg d\mu : g \in \Sigma^+, \| g \|_{L^{p',q'}} \leq 1 \right\} =: M_{p,q}(f),
\]

where \( \Sigma^+ \) is the set of all non-negative simple functions with support of finite measure.

**Proof.** As before, the \( \gtrsim_{p,q} \) inequality follows from Hölder’s inequality for Lorentz spaces. Also, if \( q = \infty \), then the result follows from Theorem 3.2. We will thus prove the inequality \( \lesssim_{p,q} \) restricted to \( q < \infty \).

Homogeneity allows us to normalize \( \| f \|_{L^{p,q}} = 1 \), we thus have to show that

\[
M_{p,q}(f) \gtrsim_{p,q} 1.
\]

Using Theorem 2.2 we can decompose \( f \) as

\[
f = \sum_{m \in \mathbb{Z}} f_m,
\]

where each \( f_m \) is a quasi-step function of height \( 2^m \) and width \( W_m \), and such that

\[
\| 2^m W_m^{1/p} \|_{L^q} \sim 1.
\]

Observe that it follows from the proof of Theorem 2.2 that each \( f_m \) must also be a simple function and, also, that \( f_m \) is 0 for \( |m| \) sufficiently large.

Define the function \( g \) by

\[
g = \sum_{m \in \mathbb{Z}} g_m
\]
where \( g_m = a_m^r f_m^{p-1}, \ a_m = 2^m W_m^{1/p} \) and \( r = \max(0, q-p) \). Clearly \( g \in \Sigma^+ \), and furthermore, since the \( f_m \)'s have disjoint supports \( E_m \):

\[
\int_X fg \, d\mu = \int_X \sum_m a_m^r f_m^p \, d\mu = \sum_m a_m^r \int_X f_m^p \, d\mu.
\]

The functions \( f_m \) are quasi-step functions of height \( 2^m \) and width \( W_m \), thus

\[
\int_X f_m^p \, d\mu \sim_p 2^{mp} W_m = a_m^p,
\]

that is

\[
\int_X fg \, d\mu \sim_p \sum_{m \in \mathbb{Z}} a_m^{r+p} \overset{(*)}{\leq} \|a_m\|_{\ell^q}^{r+p} \sim_{p,q} \|f\|_{L^{p,q}} = 1,
\]

where in \((*)\) we have used that \( \|\cdot\|_{\ell^p_1} \leq \|\cdot\|_{\ell^p_2} \) when \( p_1 \geq p_2 \) and that \( r+p \geq q \). So we only have to show that

\[ \|g\|_{L^{p',q'}} \leq 1. \]

Observe that

\[ g_m \lesssim_{p,q} a_m^r 2^{m(p-1)} \|f\|_{E_m} \]

with \( \mu(E_m) = W_m \), so a direct application of Theorem 2.2 would yield the result. The sequence \( a_m^r 2^{m(p-1)} \), however, can fail to be of lower exponential growth (at least a priori). We can fix this by defining

\[ \gamma_m = 2^m \frac{r+1}{p+1} \sup_{k \leq m} a_k^r 2^{k \frac{r+1}{p+1}} = 2^m (p-1) \sup_{k \geq 0} a_{m-k}^r 2^{-k \frac{r+1}{p+1}}. \]

Obviously \( \gamma_m \geq a_m^r 2^{m(p-1)} \), and furthermore:

\[ \gamma_{m+1} = 2^m \frac{r+1}{p+1} \sup_{k \leq m+1} a_k^r 2^{k \frac{r+1}{p+1}} \]

\[ \geq 2^m \frac{r+1}{p+1} \sup_{k \leq m} a_k^r 2^{k \frac{r+1}{p+1}} = 2 \gamma_m, \]

that is, \( \{\gamma_m\}_{m \in \mathbb{Z}} \) is of sub-exponential growth with constant \( C_1 = 2 \frac{r+1}{p+1} > 1 \). Now we can use Theorem 2.2 and the remark that follows it to reduce the problem to showing

\[ \|\gamma_m W_m^{1/p} \|_{\ell^{p'}} \lesssim_{p,q} 1. \]

But \( W_m = 2^{-mp} a_m^p \), so

\[ \|\gamma_m W_m^{1/p'} \|_{\ell^{p'}} = \|\gamma_m a_m^{p-1} 2^{-m(p-1)} \|_{\ell^{p'}} \]

\[ = \|a_m^{p-1} 2^{-m(p-1)} \|_{\ell^{p'}} \sup_{k \geq 0} a_{m-k}^r 2^{-k \frac{r+1}{p+1}} \]

\[ = \|\sup_{k \geq 0} a_{m-k}^r 2^{-k \frac{r+1}{p+1}} \|_{\ell^{p'}} \]

\[ \leq \sum_{k \geq 0} a_{m-k}^r 2^{-k \frac{r+1}{p+1}} \]

\[ \leq \sum_{k=0}^{\infty} 2^{-k \frac{r+1}{p+1}} \|a_m^{p-1} a_{m-k}^r \|_{\ell^{p'}}. \]
If \( q \leq p \Rightarrow r = 0 \), but then \( q' \geq p' \Rightarrow \| \cdot \|_{\ell^{q'}} \leq \| \cdot \|_{\ell^{p'}} \), so
\[
\|a_m^{p-1}\|_{\ell^{q'}} \leq \|a_m^{p-1}\|_{\ell^{p'}} = \|a_m\|_{\ell^p}^{p-1} = \|2^m W_{m/p}^{1/p}\|_{\ell^{p'}}^{p-1} \lesssim_{p,q} 1,
\]
which completes the proof if \( q \leq p \).

If \( q > p \), then \( r = q - p \) and
\[
\frac{p-1}{q} + \frac{q-p}{q} = \frac{1}{q'},
\]
so using Hölder’s inequality:
\[
\|a_m^{p-1}a_{m-k}^{q-p}\|_{\ell^{q'}} \leq \|a_m^{p-1}\|_{\ell^{p'}} \|a_{m-k}^{q-p}\|_{\ell^{q'}}^{q/p}
\]
\[
= \|a_m\|_{\ell^{p'}}^{p-1} \|a_{m-k}\|_{\ell^{q'}}^{q-p} \lesssim_{p,q} 1.
\]
This finishes the proof.

We can now strengthen this theorem to eliminate the qualitative assumptions of being simple:

**Corollary 3.4.** Let \( 1 < p < \infty \) and \( 1 \leq q \leq \infty \), then
\[
\|f\|_{L^{p,q}} \sim_{p,q} \sup_{\Sigma} \left\{ \int_X fg \, d\mu : g \in \Sigma, \|g\|_{L^{p',q'}} \leq 1 \right\}
\]
for all \( f \in L^{p,q} \) and where \( \Sigma \) is the set of all finite combinations of characteristic functions of sets of finite measure. Furthermore, if the measure space is \( \sigma \)-finite and if \( f \) is not a priori in \( L^{p,q} \) but the right hand side is well defined and finite, then the left hand side is also finite and we have (13).

**Proof.** We will only show this in the case \( f \in L^{p,q} \) and \( f \geq 0 \), the complete proof follows the same pattern as in the proof of Theorem 3.2.

From Hölder’s inequality we can reduce to showing only the \( \lesssim_{p,q} \) inequality. Approximate (a.e.) \( f \) by an increasing sequence \( \{f_n\}_{n \in \mathbb{N}} \) of positive simple functions of finite support (this can be achieved if the measure space is \( \sigma \)-finite or if \( f \) is a priori in \( L^{p,q} \), we omit the details).

Using the Monotone Convergence Theorem for Lorentz spaces 1.8:
\[
\|f\|_{L^{p,q}} = \sup_n \|f_n\|_{L^{p,q}} \sim_{p,q} \sup_n \sup_{\Sigma^+} \left\{ \int_X f_n g \, d\mu : g \in \Sigma^+, \|g\|_{L^{p',q'}} \leq 1 \right\}
\]
\[
= \sup \left\{ \sup_n \int_X f_n g \, d\mu : g \in \Sigma^+, \|g\|_{L^{p',q'}} \leq 1 \right\}
\]
\[
= \sup \left\{ \int_X f g \, d\mu : g \in \Sigma^+, \|g\|_{L^{p',q'}} \leq 1 \right\},
\]
where we have used the (usual) monotone convergence theorem in the last line. □

This shows that there is a norm (the one defined using the dual characterization) which is equivalent to the \( L^{p,q} \) quasi-norm defined at the beginning when \( 1 < p < \infty \).

4. **Marcinkiewicz Interpolation Theorem**

In this section we will present a proof of the Marcinkiewicz interpolation theorem using the decomposition introduced in section 2.

We will use duality from the outset, indeed, we will need the following extension of Theorem 3.2:
Theorem 4.1. Let \((X, \mu)\) be a measure space and let \(0 < p < \infty\), the following statements are equivalent for \(f \in L^{p, \infty}\):

1. \(\|f\|_{L^{p, \infty}} \lesssim p^{1/2}\).
2. For every set \(E\) of finite measure there exists a subset \(E' \subseteq E\) such that \(\mu(E) \geq \frac{1}{2} \mu(E)\) such that
\[
\left| \int_X f \mathbb{1}_{E'} \, d\mu \right| \lesssim p \mu(E')^{1/p'}.
\]

Furthermore, the theorem holds without the a priori assumption of \(f \in L^{p, \infty}\) provided \((X, \mu)\) is \(\sigma\)-finite.

Proof. We will assume \(0 < p \leq 1\), we will also assume \(f \geq 0\) and \(\|f\|_{L^{p, \infty}} = 1\) without loss of generality. The proof of (2) \(\Rightarrow\) (1) is almost identical to the one in Theorem 3.2, let us show the other one.

Let \(E \subset X\) be a set of finite (and positive) measure and let \(\alpha = \left(\frac{2}{\mu(E)}\right)^{1/p}\). Define \(E' = E \setminus \{x : f(x) > \alpha\}\), then, since we have the estimate \(\mu(\{x : f(x) > \alpha\}) \leq \frac{\mu(E)}{2}\), we can bound \(\mu(E) \geq \mu(E') \geq \frac{1}{2} \mu(E)\). Now we can estimate the integral
\[
\int_X f \mathbb{1}_{E'} \, d\mu \leq \left(\frac{2}{\mu(E)}\right)^{1/p} \mu(E') \leq 2^{1/p} \mu(E')^{1/p'}.
\]

The Marcinkiewicz interpolation theorem, as in complex interpolation, uses the boundedness of an operator in two “extremal spaces” to provide boundedness in the “intermediate spaces”. However, while complex interpolation of \(L^p\) spaces requires the extremal spaces to also be \(L^p\) spaces, the Marcinkiewicz Interpolation Theorem only needs them to be weak-\(L^p\) spaces. It actually requires a bit less, the following definitions will make the exposition easier:

Definition 4.2. Let \(T\) be an operator defined on a subset of the set of measurable functions of a measure space \((X, \mu)\) and that takes values in the measurable functions of another measure space \((Y, \nu)\).

- We will say that \(T\) is of strong type \((p, q)\) if \(T\) extends to a bounded operator from \(L^p(X, \mu)\) to \(L^q(Y, \nu)\).
- We will say that \(T\) is of weak type \((p, q)\) if \(T\) extends to a bounded operator from \(L^p(X, \mu)\) to \(L^{q, \infty}(Y, \nu)\).

In what follows we will always assume \(T\) to be an operator defined on a set \(D\) closed under addition, multiplication by scalars and containing \(\Sigma_c(X, \mu)\): the set of finite linear combinations of characteristic functions of sets of finite measure of \((X, \mu)\).

We will call \(T\) sublinear if
\[
|T(f + g)| \leq |T(f)| + |T(g)|, \quad |T(\lambda f)| = |\lambda| |T(f)|
\]
for all \(f, g\) in the domain of \(T\) and \(\lambda \in \mathbb{C}\).

We say that a sublinear operator \(T\) is of restricted weak type \((p, q)\) if
\[
\|Tf\|_{L^{q, \infty}} \lesssim HW^{1/p}
\]
for all sub-step functions \(f \in D\) of height \(H\) and width \(W\).
The next result shows that, usually, an operator which is of restricted weak type on characteristic functions is also of restricted weak type on $\Sigma_c$. More precisely:

**Proposition 4.3.** Let $0 < p \leq \infty$, $0 < q \leq \infty$, $A > 0$ and $T$ be a sublinear operator defined on $\Sigma_c$. Then the following statements are equivalent:

1. $T$ satisfies
   \[ \|Tf\|_{L^{q,\infty}} \lesssim AHW^{1/p} \]
   for all sub-step functions $f$ of height $H$ and width $W$ in $\Sigma_c$.
2. For every set $F \subseteq Y$ of finite measure there exists a subset $F' \subseteq F$ with $\nu(F') \geq \frac{1}{2}\nu(F)$ and such that for all $E \subseteq X$ of finite measure
   \[ \int_Y |T\mathbb{1}_E|\mathbb{1}_{F'} \, d\nu \lesssim A\mu(E)^{1/p}\nu(F')^{1/q'}. \]

**Proof.** That (1) $\implies$ (2) is an easy consequence of Theorem 4.1 and Hölder’s inequality for Lorentz spaces, so we will prove the reverse implication.

First assume that $f$ takes the form
\[ f_N = \sum_{j=1}^N 2^{-j}\mathbb{1}_{E_j} \]
where $\mu(E_j) \leq W$. In this case $\|Tf_N\|_{L^{q,\infty}} \lesssim AW^{1/p}$ (the constant being independent of $N$). To show this first observe that using sublinearity and Lemma 5.1 it suffices to show

\[ (15) \quad \|T\mathbb{1}_{E_j}\|_{L^{q,\infty}} \lesssim AW^{1/p}. \]

Now let $F \subseteq Y$ be a set of finite measure. By hypothesis there exists a subset $F' \subseteq F$ which satisfies $\nu(F') \geq \frac{1}{2}\nu(F)$ and
\[
\int_Y |T\mathbb{1}_{F'}\mathbb{1}_{F_j} \, d\nu \lesssim A\mu(E_j)^{1/p}\nu(F')^{1/q'} \\
\leq \nu(F')^{1/q'} A W^{1/p}.
\]

So by Theorem 4.1 we obtain (15).

Now let $0 \leq f \leq 1$ be an arbitrary function in $\Sigma_c$ of width $W$:
\[ f = \sum_{j=1}^M a_j \mathbb{1}_{E_j}, \]
the $E_j$’s disjunctly supported. If $d_j(x)$ denotes the $j$-th digit in the binary expansion of $f(x)$ and we define
\[ f_N = \sum_{j=1}^N 2^{-j}d_j = \sum_{j=1}^N 2^{-j}\mathbb{1}_{F_j}, \]
then we easily see that
\[ f - f_N = \sum_{j=1}^M b_{j,N}\mathbb{1}_{E_j}. \]
with $0 \leq b_{j,N} \leq 2^{-N}$. So:

$$\|Tf\|_{L^{q},\infty} \lesssim \|Tf_N\|_{L^{q},\infty} + \|T(f - f_N)\|_{L^{q},\infty}$$

$$\lesssim AW^{1/p} + \left\| \sum_{j=1}^{M} b_{j,N} |T1_{E_j}| \right\|_{L^{q},\infty},$$

since the second term can be made arbitrarily small we conclude that

$$\|Tf\|_{L^{q},\infty} \lesssim AW^{1/p}.$$

By homogeneity and sublinearity we obtain the same bound (with a possibly larger constant) for arbitrary $f \in \Sigma_c$.

To prove the Marcinkiewicz interpolation theorem we first need a weaker result:

**Proposition 4.4.** Let $0 < p_0, p_1, q_0, q_1 \leq \infty$, $A_0, A_1 > 0$ and suppose that $T$ is a sublinear operator defined on $\Sigma_c$ which is of restricted weak type $(p_i, q_i)$ with constants $A_i$ for $i = 0, 1$.

Then it is of restricted weak type $(p_\theta, q_\theta)$, i.e.:

$$\|Tf\|_{L^{q_\theta},\infty} \lesssim p_0^{1/q_\theta} q_0^{1/q_\theta} p_1^{1/q_1} q_1^{1/q_1} A_{\theta} \mu(E)^{1/p_\theta} \nu(F)^{1/q_\theta},$$

where

$$\frac{1}{p_\theta} = \frac{1 - \theta}{q_0} + \frac{\theta}{q_1}, \quad \frac{1}{q_\theta} = \frac{1 - \theta}{q_0} + \frac{\theta}{q_1},$$

and

$$A_{\theta} = A_0^{1-\theta} A_1^\theta.$$

**Proof.** Theorem 4.3 reduces matters to showing that for every set $F \subseteq Y$ of finite measure there exists a subset $F' \subseteq F$ with $\nu(F') \geq \frac{1}{2} \nu(F)$ and such that for all sets $E \subseteq X$ of finite measure:

$$\int_Y 1_{F'} 1_{E} d\nu \lesssim p_0^{1/q_0} p_1^{1/q_1} A_{\theta} \mu(E)^{1/p_\theta} \nu(F)^{1/q_\theta},$$

but, using Theorem 4.3 and the hypotheses, we have that there exists a subset $F'$ of $F$ with $\nu(F') \geq \frac{1}{2} \nu(F)$ and that for all sets $E$ of finite measure

$$\int_Y 1_{F'} 1_{E} d\nu \lesssim p_0^{1/q_0} p_1^{1/q_1} A_{\theta} \mu(E)^{1/p_\theta} \nu(F')^{1/q_\theta},$$

for $i = 0, 1$. The result now follows from the trivial scalar interpolation inequality:

$$X \leq Y \quad \land \quad X \leq Z \implies X \leq Y^{1-\theta} Z^\theta$$

for $X, Y, Z \geq 0$ and $0 \leq \theta \leq 1$, and an application of Theorem 4.3 once more.

We need one last lemma:

**Lemma 4.5.** Let

$$\Lambda_\lambda(x, y) = (1 - \lambda)x + \lambda y,$$

then

$$\Lambda_\alpha(\Lambda_\beta(x, y), \Lambda_\gamma(x, y)) = \Lambda_{\alpha (\beta, \gamma)}(x, y).$$

We are now ready to state and prove the main theorem of this section:
Theorem 4.6. Let $T$ be a sublinear operator defined on $\Sigma^+$. Suppose that $0 < p_i, q_i \leq \infty$ and $q_0 \neq q_1$. If $T$ satisfies
\[ \|Tf\|_{L^{q_i,r}} \lesssim A_i \|f\|_{L^{p_i,1}} \quad i = 0, 1, \]
for all $E$ of finite measure, then for all $1 \leq r \leq \infty$ and $0 < \theta < 1$ such that $q_0 > 1$ we have
\[ \|Tf\|_{L^{q_0,r}} \lesssim c_0 p_0, q_0, p_1, q_0, r, \theta A_0 \|f\|_{L^{p_0,r}} \]
for all $f$ in $\Sigma_c$.

Proof. Using Theorem 4.3 we can assume that $T$ is of restricted weak type $(p_i, q_i)$ with constants $A_i$.

We will suppose first that $q_i \geq 1$. Let $f \in \Sigma^+$, which by homogeneity we may assume to be normalized so that $\|f\|_{L^{p_0,r}} = 1$. By duality (Theorem 3.3), it will be enough to verify the estimate
\[ \int_Y |Tf|g d\nu \lesssim_{p_0, q_0, p_1, q_1, r, \theta} A_0 \]
for all $g \in \Sigma^+$ with $\|g\|_{L^{q_0,r}} = 1$. In what follows all constants will be assumed to depend on $p_0, q_0, p_1, q_1, r$ and $\theta$ and will be omitted.

By hypothesis and Hölder’s inequality we have
\[ \int_Y |Tu|v d\nu \lesssim A_i HH' W^{1/p_i} W'^{1/q_i} \leq A_i HH' \min_{i=0,1} W_i^{1/p_i} W'^{1/q_i} \]
for all sub-step functions $u \in \Sigma^+(X, \mu)$ of height $H$ and width $W$ and for all sub-step functions $v \in \Sigma^+(Y, \nu)$ of height $H'$ and width $W'$.

By Theorem 2.2 we can decompose $f$ and $g$ as
\[ f = \sum_m f_m \quad \text{and} \quad g = \sum_n g_n \]
where each $f_m$ is a sub-step function of height $H_m$ and width $2^m$ and where each $g_n$ is a sub-step function of height $H'_n$ and width $2^n$. These decompositions also satisfy
\[ \|H_m 2^{m/p_o}\|_{L^r} \sim \|H'_n 2^{n/q_o}\|_{L^r} \sim 1. \]
Recall that, since $f$ and $g$ are simple, only a finite number of $f_m$’s and $g_n$’s are non-zero.

Using (17) and sublinearity we have
\[ \int_Y |Tf|g d\nu \lesssim A_0 \sum_{m,n} H_m H'_n \min_{i=0,1} (2^{m/p_i} 2^{n/q_i}) \]
Call $a_m = H_m 2^{m/p_o}$ and $b_n = H'_n 2^{n/q_o}$, we then have to obtain the estimate
\[ \sum_{m,n} a_m b_n \min_{i=0,1} (2^{m(1/p_i-1/p_o)} 2^{n(1/q_i-1/q_o)}) \lesssim 1, \]
where $\|a_m\|_{L^r} = \|b_n\|_{L^r} = 1$.

After reorganizing terms we can write the third factor as
\[ \min \left( m \left( \frac{1}{p_0} - \frac{1}{p_1} \right) + n \left( \frac{1}{q_0} - \frac{1}{q_1} \right), -m \left( \frac{1}{p_0} - \frac{1}{p_1} \right) - n \left( \frac{1}{q_1} - \frac{1}{q_0} \right) \right), \]
which can be further simplified to
\[ \min \left( m - \beta, -m + \beta \right), \]
where $\alpha = \frac{1}{p_0} - \frac{1}{p_1}$ and $\beta = \frac{1}{q_0} - \frac{1}{q_1}$, which is easily seen to be true. Let
\[ \varphi(m - \gamma) = \min \left( \alpha m - \gamma, -\alpha m + \gamma \right) \]
...
where \( \gamma = \frac{2}{\alpha} \), then we have to show that
\[
\sum_{m,n} a_{m,n} \varphi(m - \gamma n) \lesssim 1.
\]
We can change variables and let \( m \mapsto m + \lfloor \gamma n \rfloor \) and reduce to showing
\[
\sum_{m,n} a_{m+\lfloor \gamma n \rfloor} b_n \varphi(m + \lfloor \gamma n \rfloor - \gamma n) \lesssim 1.
\]
Observe that \( |\lfloor \gamma n \rfloor - \gamma n| \leq 1 \), so we easily see that
\[
\varphi(m + \lfloor \gamma n \rfloor - \gamma n) \sim \varphi(m),
\]
thus we end up with
\[
\sum_m \varphi(m) \sum_n a_{m+\lfloor \gamma n \rfloor} b_n \lesssim \left\| \sum_n a_{m+\lfloor \gamma n \rfloor} b_n \right\|_{\ell_1^\infty}
\]
since \( \varphi \) is in \( \ell^1 \). Applying Hölder’s inequality with exponents \( r \) and \( r' \) we reduce to showing that
\[
\left\| a_{m+\lfloor \gamma n \rfloor} \right\|_{\ell_1^r} \lesssim 1
\]
uniformly in \( m \), but this is just the observation that the sum of \( a'_{m+\lfloor \gamma n \rfloor} \) is bounded by a constant depending on \( \gamma \) times the sum of \( a'_{m+n} \), which is of course bounded by a constant independent of \( m \).

Finally, we show how to drop the assumption of \( q_i > 1 \). Observe that we may interpolate using Theorem 4.4 to obtain restricted weak type \((p_0, q_0)\) boundedness of \( T \) for \( 0 \leq \theta \leq 1 \). Since we are imposing \( q_0 > 1 \), we may assume that \( q_i > 1 \) for at least one \( i \in \{0, 1\} \) (otherwise the theorem is void), thus, there must be a \( \tau \in (0, 1) \) such that \( 1 < q_\tau < q_0 \) and such that \( T \) is of restricted weak type \((p_\tau, q_\tau)\). We can now interpolate using what we have just proved from \((p_\tau, q_\tau)\) to the \((p_i, q_i)\) which has \( q_i > 1 \) and use Lemma 4.5 to ensure that \((p_\theta, q_\theta)\) is in between \((p_\tau, q_\tau)\) and \((p_i, q_i)\).

\[
\square
\]

Corollary 4.7 (Marcinkiewicz). Let \( T \) be a sublinear operator that is of weak type \((p_i, q_i)\) with constants \( A_i > 0 \) and such that \( 1 \leq p_i \leq p_i \leq \infty \) for \( i \in \{0, 1\} \), suppose also that \( q_0 \neq q_1 \). Then, for \( 0 < \theta < 1 \), \( T \) is of strong type \((p_\theta, q_\theta)\) with constant
\[
\frac{1}{p_\theta} = \frac{1 - \theta}{p_0} + \frac{\theta}{p_1} \quad \text{and} \quad \frac{1}{q_\theta} = \frac{1 - \theta}{q_0} + \frac{\theta}{q_1}.
\]

Proof. One just needs to recall that the \( L^{p,q} \) are nested and that \( L^{p,p} \) coincides with \( L^p \), we leave the easy details to the reader. \( \square \)

5. Appendix

Lemma 5.1. Let \( X \) be a vector space endowed with a quasi-norm \( \| \cdot \| \), that is, there exists a constant \( C_1 > 0 \) such that
\[
\| f + g \| \leq C_1(\| f \| + \| g \|).
\]
Then, for any sequence \( \{f_n\}_{n \in \mathbb{N}} \) of elements of \( X \) verifying
\[
\| f_j \| \lesssim AC_2^{-j}
\]
for some \( A > 0 \) and \( C_2 > 1 \), we have
\[
\left\| \sum_{j=1}^N f_j \right\| \leq AC_4
\]
where $C_3$ does not depend on $N$ or $A$.

**Proof.** Choose $\kappa \in \mathbb{N}$ so large that $C_1 C_2^{-\kappa} < 1$ and write $N$ as

$$N = \kappa d + r$$

where $d \in \mathbb{N}$ and $0 \leq r < \kappa$. Observe that $\kappa$ does not depend on $N$.

We can now split the sum as follows

$$\sum_{j=1}^{N} f_j = \sum_{j=1}^{\kappa d} f_j + \sum_{j=1}^{r} f_{kd+j}.$$ 

The norm of the second sum is trivially bounded by $A$ times some constant depending on $C_1$ and $C_2$ (since $r < \kappa$), so we can center our efforts on the first sum. To this end we factorize the sum as

$$\sum_{j=1}^{\kappa d} f_j = \sum_{m=0}^{d-1} \sum_{j=1}^{\kappa} f_{km+j}.$$ 

Now observe that, by induction, we have the following estimate:

$$\left\| \sum_{j=1}^{n} g_j \right\| \leq \sum_{j=1}^{n} C_1^j \left\| g_j \right\|$$

for any sequence $\{g_j\}$ of elements of $X$. Using this we can estimate the first sum as follows:

$$\left\| \sum_{j=1}^{\kappa d} f_j + \sum_{j=1}^{r} f_{kd+j} \right\| \leq \sum_{m=0}^{d-1} \sum_{j=1}^{\kappa} C_1^m C_2^{-(\kappa m+j)}$$

$$= A \sum_{m=0}^{d-1} (C_1 C_2^{-\kappa})^m \sum_{j=1}^{\kappa} C_1^{j+1} C_2^{-j}$$

$$\leq A \sum_{m=0}^{\infty} (C_1 C_2^{-\kappa})^m \sum_{j=1}^{\kappa} C_1^{j+1} C_2^{-j}$$

$$\lesssim C_1, C_2 \ A \sum_{j=1}^{\kappa} C_1^{j+1} C_2^{-j}$$

$$\lesssim C_1, C_2 \ A.$$ 

$\square$

**References**


