Calculus III - Quiz 9 - Spring 2015

April 20, 2015

Name: ______________________________

In this quiz you will find a formula for the area of the ellipse given by the equation

$$E = \{(x, y) : \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 \leq 1\}.$$

1. (4 points) Parametrize the boundary of E (positively oriented) by a vector function $r(t)$ with $t \in [0, 2\pi]$.

> **Solution:** The simplest parametrization is
>
> $$r(t) = \langle a \cos t, b \sin t \rangle.$$

2. (2 points) Find *any* vector field $F = \langle P, Q \rangle$ such that

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1.$$

Note: the simplest the F the better, try to come up with one for which P is 0, for example.

> **Solution:** One possible F is
>
> $$F(x, y) = \langle 0, x \rangle.$$

3. (4 points) Compute

$$\text{Area} = \iint_E 1 \, dA$$

using Green's theorem (any other method will receive no credit).

Solution: Using the F that we found in part (2) we can write:

$$\begin{aligned}
\iint_E 1\, dA &= \int_{\text{Boundary of } E} F \cdot dr \\
&= \int_0^{2\pi} 0 \cdot (-a \sin t) + (a \cos t)(b \cos t)\, dt \\
&= \int_0^{2\pi} ab \cos^2 t\, dt \\
&= ab \int_0^{2\pi} \cos^2 t\, dt \\
&= ab \int_0^{2\pi} \frac{1 + \cos(2t)}{2}\, dt \\
&= ab\Big(\pi + \Big[\frac{\sin(4t)}{4}\Big]_0^{2\pi}\Big) \\
&= \pi ab.
\end{aligned}$$