Calculus III - Quiz 4 - Spring 2015

March 19, 2015

Name: ______________________________

Let f denote a function defined on $\mathbb{R}^2$ which has derivatives of all orders at all points and suppose that $(0,0)$ is a critical point. We saw in class a way to determine whether the $(0,0)$ was a local minimum, maximum, or saddle point. One first had to compute the Hessian of f at $(0,0)$:

$$\text{Hess}\, f(0,0) = \begin{pmatrix} f_{xx}(0,0) & f_{xy}(0,0) \\ f_{yx}(0,0) & f_{yy}(0,0) \end{pmatrix}.$$

Then there were several possibilties depending on the sign of f_{xx} and of the determinant $D = f_{xx}(0,0)f_{yy}(0,0) - f_{xy}(0,0)^2$. If $D < 0$ then the point is a saddle point. If $D > 0$ and $f_{xx}(0,0) > 0$ then the point is a local minimum and if $D > 0$ and $f_{xx}(0,0) < 0$ then the point is a local maximum.

So what happens when $D = 0$? In this quiz you will fill the gap by studying the special case of $D = 0$ and $f_{xx}(0,0) > 0$.

To do this we will consider a line that goes through $(0,0)$, we can assume it's a straight line:

$$\mathbf{r}(t) = \langle at, bt \rangle.$$

What does f look like along this line? That is, what is the shape of $(at, bt, f(\mathbf{r}(t)))$? If the shape of this curve is always concave up, then the point should be a local minimum, and if it is always concave down then it should be a local maximum. You will end-up showing that, under the above conditions, the curve is always concave up.

1. (4 points) Let us denote by $g(t)$ the function defined by

$$g(t) = f(\mathbf{r}(t)).$$

Assume that $(0,0)$ is a critical point of f, i.e.: $\nabla f(0,0) = \langle 0,0 \rangle$.

1. Show that $g'(0) = 0$.
2. Show that $g''(0) = a^2 f_{xx}(0,0) + 2ab f_{xy}(0,0) + b^2 f_{yy}(0,0)$.

Hint: Use the chain rule.

Solution: Using the chain rule once we get

$$g'(0) = f_x(0,0)a + f_y(0,0)b,$$

but this is 0 since $(0,0)$ is a critical point for f.

To find $g''(0)$ we just apply the chain rule again, but with more care since the derivatices f_x and f_y are also functions of x and y:

$$g''(0) = \frac{d}{dt}\bigg|_{t=0} af_x(x,y) + bf_y(x,y),$$

where $x = at$ and $y = bt$.

By the chain rule, this becomes:

$$g''(0) = a^2 f_{xx}(0,0) + abf_{xy}(0,0) + baf_{yx}(0,0) + b^2 f_{yy}(0,0),$$

and since $f_{xy} = f_{yx}$, this yields the claim.

2. (6 points) Now assume, in addition to the above, that:

$$f_{xx}(0,0) > 0 \quad \text{and} \quad D = f_{xx}(0,0)f_{yy}(0,0) - f_{xy}(0,0)^2 = 0.$$

Show that $g''(0) \geq 0$. *Hint: complete the square and use that* $D = 0$

Solution: From the equation, we see that

$$f_{xy}^2(0,0) = f_{xx}(0,0)f_{yy}(0,0),$$

so $f_{xy}(0,0) = \pm\sqrt{f_{xx}(0,0)f_{yy}(0,0)}$.

Now we can complete the square:

$$\begin{aligned} g''(0) &= a^2 f_{xx}(0,0) \pm 2ab\sqrt{f_{xx}(0,0)f_{yy}(0,0)} + b^2 f_{yy}(0,0) \\ &= (a\sqrt{f_{xx}(0,0)} \pm b\sqrt{f_{yy}(0,0)})^2. \end{aligned}$$

Since squares are always nonnegative, the claim is proved.