Math 2374 - Quiz 9

Name: ______________________________

Section: ______

Consider the function

$$f(x, y) = (x^2 + y^2)^2 - (x^2 + y^2).$$

The gradient of f is

$$\nabla f(x, y) = \big(2x(2(x^2 + y^2) - 1), 2y(2(x^2 + y^2) - 1)\big)$$

And the Hessian is:

$$\text{Hess}\, f(x, y) = \begin{pmatrix} 4(x^2 + y^2) + 8x^2 - 2 & 8xy \\ 8xy & 4(x^2 + y^2) + 8y^2 - 2 \end{pmatrix}$$

1. Show that $(0, 0)$ is a critical point, evaluate f there, and classify it.

2. Prove that the critical points of f are the point $(0, 0)$ and all the points in the circle $x^2 + y^2 = \frac{1}{2}$. You can argue as follows: if $x^2 + y^2 \neq \frac{1}{2}$ then..., and if $x^2 + y^2 = \frac{1}{2}$ then...

3. Prove that the Hessian of f at every point in $x^2 + y^2 = \frac{1}{2}$ has 0 determinant.

4. Since the Hessian of f has zero determinant in that circle we cannot use the second derivative test. However, we can still compute the global minimum of f. To do this first observe that f is symmetric under rotations (in fact, it depends only on the radius). So finding the minimum of f is equivalent to finding the minimum of

$$g(t) = f(t, 0).$$

Compute the global minimum of g.