Math 2374 - Quiz 6

Name: ____________________________________

Section: _______

Let S be the portion of the plane

$$x + y + z = 1$$

lying in the first octant ($x, y, z \geq 0$).

1. Find a parametrization Φ of S specifying the domain of Φ.

Solution: We can represent this surface as a graph:

$$z = 1 - y - z,$$

so $\Phi : D \to \mathbb{R}^3$ can be defined by $\Phi(u, v) = (u, v, 1 - u - v)$. The domain of Φ is the one that will map to that portion of the plane, which is a triangle:

$$D = \{(x, y) : 0 \leq x \leq 1 \quad \text{and} \quad 0 \leq y \leq 1 - x\}$$

2. Find a non-zero vector *orthogonal* to the surface at the point $(1/3, 1/3, 1/3)$.

Solution: There are two ways of doing this: one if using the fact that this is a plane and we know how to find normals to planes from the beginning of the course (just take the coefficients of x, y, and z in order). But since we just found a parametrization we can use that as well:

$$w = T_u \times T_v,$$

where

$$T_u = (1, 0, -1) \quad \text{and} \quad T_v = (0, 1, -1),$$

so

$$w = (1, 0, -1) \times (0, 1, -1) = (1, 1, 1).$$

In theory we now have to evaluate this at the (u, v) whic map to $(1/3, 1/3, 1/3)$, but of course this is a constant vector so we don't have to do that.

3. Compute the surface integral

$$\iint_S x\, dS.$$

Solution: We have computed $T_u \times T_v$ before, so now we can just compute its magnitude

$$\|T_u \times T_v\| = \|(1,1,1)\| = \sqrt{3}.$$

So

$$\begin{aligned}\iint_S x\, dS &= \iint_D u\sqrt{3}\, du\, dv \\ &= \int_0^1 \int_0^{1-u} u\sqrt{3}\, dv\, du \\ &= \sqrt{3}\int_0^1 u(1-u)\, du \\ &= \sqrt{3}\int_0^1 u - u^2\, du \\ &= \sqrt{3}(1/2 - 1/3) \\ &= \frac{\sqrt{3}}{6}.\end{aligned}$$