Math 2374 - Quiz 5

Name: ____________________________________

Section: ______

Consider the vector field

$$V(x, y) = (2xy^3 + 2x, 3x^2y^2 + \cos y)$$

1. Show that the *scalar curl* is 0. Remember that the scalar curl is the z-component of the curl of the vector field W given by

$$W(x, y, z) = (2xy^3 + 2x, 3x^2y^2 + \cos y, 0).$$

Solution: The scalar curl is $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$. One easily sees that $\partial Q/\partial x = 6xy^2 = \partial P/\partial y$, so

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 6xy^2 - 6xy^2 = 0.$$

2. Let c be a positively oriented simple closed curve in the plane. Show that

$$\int_c V \cdot ds = 0.$$

You are not given the precise form of the curve, so you may want to avoid integrating directly.

Solution: By Green's theorem we know that

$$\int_c V \cdot ds = \iint_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \, dxdy,$$

where R is the region enclosed by the c. By the above problem the integrand is 0, so we're done.

3. In the last part of the quiz we're going to find a potential f whose gradient is V. In other words, we're looking for a function f satisfying

$$V = \nabla f.$$

   1. If such an f were to exist, it would necessarily need to verify

   $$\frac{\partial f(x,y)}{\partial x} = 2xy^3 + 2x.$$

   Find a function that satisfies this. Remember to add a $+C(y)$ at the end, this way you will get *all* such functions.

   Solution: To find a function whose derivative is a given function we integrate it:

   $$f(x,y) = \int 2xy^3 + 2x\,dx = x^2y^3 + x^2 + C(y).$$

   2. In the previous question you found *all* functions whose partial derivative with respect to x is the first component of V, they have the form

   $$f(x,y) = \text{Something} + C(y).$$

   You want f to *also* satisfy

   $$\frac{\partial f(x,y)}{\partial y} = 3x^2y^2 + \cos y.$$

   Plug the expression of f that you found above into this equation in order to see what C needs to look like, and then find a C which works.

   Solution: Plugging the expression above we have

   $$\frac{\partial}{\partial y}(x^2y^3 + x^2 + C(y)) = 3x^2y^2 + \cos y.$$

   The left hand side is

   $$3x^2y^2 + C'(y),$$

   so the term $3x^2y^2$ cancels and we get that $C'(y) = \cos y$.
   We just need to find a function C that satisfies $C'(y) = \cos y$, so $C(y) = \sin y$.
   Putting it all together we get that a possible f is

   $$f(x,y) = x^2y^3 + x^2 + \sin y.$$