Math 2374 - Quiz 4

Name: ____________________

Section: ______

Suppose that a (unit mass) particle moves along a path $c(t) = (x(t), y(t))$ in a force field according to Newton's law:

$$F(c(t)) = c''(t),$$

and suppose that the F is a gradient field: there exists a function U such that $\nabla U = F$.

1. Show that

$$\frac{d}{dt}\frac{1}{2}\|c'(t)\|^2 = c'(t) \cdot \nabla U(c(t))$$

Solution: By the chain rule we have

$$\begin{aligned}\frac{d}{dt}\frac{1}{2}\|c'(t)\|^2 &= \frac{1}{2}\frac{d}{dt}(x'(t)^2 + y'(t)^2)\\ &= \frac{1}{2}(2x'(t)x''(t) + 2y'(t)y''(t))\\ &= x'(t)x''(t) + y'(t)y''(t).\end{aligned}$$

Now, we know that $F(c(t)) = c''(t) = (x''(t), y''(t))$, so we in fact have

$$\begin{aligned}\frac{d}{dt}\frac{1}{2}\|c'(t)\|^2 &= x'(t)F_1(c(t)) + y'(t)F_2(c(t))\\ &= c'(t) \cdot F(c(t))\\ &= c'(t) \cdot \nabla U(c(t)).\end{aligned}$$

2. Prove that the energy
$$E(t) = \frac{1}{2}\|c'(t)\|^2 - U(c(t))$$
is constant.

You may use the solution to the first question. *Hint: What is $E'(t)$?*

Solution: To show that E is constant it suffices to show that the derivative is 0 everywhere.

We have already computed the first part of the derivative, in the previous part of the problem, so:
$$E'(t) = c'(t) \cdot \nabla U(c(t)) - \frac{d}{dt}\Big(U(c(t))\Big).$$

And now we can use the chain rule to compute the second part of the derivative:
$$\begin{aligned}\frac{d}{dt}\Big(U(c(t))\Big) &= \frac{\partial U}{\partial x}\frac{dx}{dt} + \frac{\partial U}{\partial y}\frac{dy}{dt} \\ &= \nabla U(c(t)) \cdot c'(t),\end{aligned}$$

which is what we were adding before, so $E'(t) = 0$.