Math 2374 - Midterm 2

Name: ______________________________

Section: ______

Student ID: ______________________________

Signature: ______________________________

- This exam consists of 6 pages (including this one) and 5 questions.
- Do not give numerical approximations to quantities such as $\sin 5$, π, or $\sqrt{2}$. However, you should simplify $\cos \pi/4 = \sqrt{2}/2$, $e^0 = 1$, and so on.
- Show your work, in a reasonably neat and coherent way, in the space provided. All answers must be justified by valid mathematical reasoning, including the evaluation of definite and indefinite integrals. To receive full credit on a problem, you must show enough work so that your solution can be followed by someone without a calculator.
- Mysterious or unsupported answers will not receive full credit. Your work should be mathematically correct and carefully and legibly written.
- A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations will receive partial credit.
- Full credit will be given only for work that is presented neatly and logically; work scattered all over the page without a clear ordering will receive from little to no credit.

1	2	3	4	5	Total

1. (20 points)

(a) (10 points) Let R be the region of $\mathbb{R}^3$ bounded by the plane $z = 0$, the cylinder $x^2+y^2 = 9$, and the plane $x + z = 4$. Express the volume of R as a triple integral, but do not evaluate it.

Solution: In cartesian coordinates:

$$\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_{0}^{4-x} 1\, dzdydx$$

In cylindrical coordinates:

$$\int_{0}^{3}\int_{0}^{2\pi}\int_{0}^{4-r\cos\theta} r\, dzd\theta dr$$

(b) (10 points) Compute the volume in the previous question using cylindrical coordinates.

Solution: Using the previous equation:

$$\begin{aligned}\int_{0}^{3}\int_{0}^{2\pi}\int_{0}^{4-r\cos\theta} r\, dzd\theta dr &= \int_{0}^{3}\int_{0}^{2\pi} r(4-r\cos\theta)\, d\theta dr\\ &= \int_{0}^{3}\int_{0}^{2\pi} 4r - r^2\cos\theta\, d\theta dr\\ &= \int_{0}^{3} 8\pi r\, dr\\ &= 36\pi.\end{aligned}$$

2. (20 points) Let γ be the curve in $\mathbb{R}^3$ given by

$$\gamma(t) = (t^2, \log t, 2t),$$

where $\log t$ is the natural logarithm.

Compute the length of the curve γ from the point $t = 1$ to $t = 2$.

Solution: We need to integrate $\|\gamma'(t)\|$ from $t = 1$ to $t = 2$. So we first compute that:

$$\begin{aligned} \|c'(t)\| &= \|(2t, 1/t, 2)\| \\ &= \sqrt{4t^2 + \frac{1}{t^2} + 4}. \end{aligned}$$

We can simplify this with a little work:

$$\begin{aligned} \sqrt{4t^2 + \frac{1}{t^2} + 4} &= \sqrt{\frac{4t^4 + 1 + 4t^2}{t^2}} \\ &= \frac{\sqrt{4t^4 + 4t^2 + 1}}{t} \\ &= \frac{\sqrt{(2t^2 + 1)^2}}{t} \\ &= 2t + \frac{1}{t}. \end{aligned}$$

Now we just need to integrate this, so

$$\int_1^2 2t + \frac{1}{t}\, dt = \left[t^2 + \log t\right]_1^2 = 3 + \log 2$$

3. (20 points) Let V be the vector field given by

$$V(x,y,z) = (yz^2, xy^2, zx^2).$$

Is this vector field conservative? If so find the potential.

Solution: First we need to compute the curl to check whether it is conservative:

$$\begin{aligned}\operatorname{curl} V = \nabla \times V = \det \begin{pmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz^2 & xy^2 & zx^2 \end{pmatrix} \\ = (0, 2zy - 2zx, y^2 - z^2).\end{aligned}$$

This is not 0, so the vector field cannot be conservative.

4. (20 points) Let γ be the curve given by

$$\gamma(t) = (t, 2\cos(2t), \sin(2t)) \quad 0 \leq t \leq \pi$$

Compute

$$\int_\gamma y\,dx + (y^2 + 4z^2)\,dy + x\,dz$$

Solution: First we compute γ':

$$\gamma'(t) = (1, -4\sin(2t), 2\cos(2t).$$

Now we can plug in the definition of the line integral:

$$\begin{aligned}
\int_\gamma y\,dx + (y^2 + 4z^2)\,dy + x\,dz &= \int_0^\pi 2\cos(2t) + (4\cos(2t)^2 + 4\sin(2t)^2)(-2\sin(2t)) + t \cdot 2\cos(2t)\,dt \\
&= \int_0^\pi 2\cos(2t) + 4(-2\sin(2t)) + t \cdot 2\cos(2t)\,dt \\
&= \int_0^\pi 2\cos(2t) - 8\sin(2t) + 2t\cos(2t)\,dt.
\end{aligned}$$

One can easily integrate the first two terms and see that they are both zero. For the last term we can use integration by parts

$$\begin{aligned}
\int_0^\pi 2t\cos(2t)\,dt &= \int_0^\pi t\big(\sin(2t)\big)'\,dt \\
&= \Big[t\sin(2t)\Big]_0^\pi - \int_0^\pi \sin(2t)\,dt \\
&= 0
\end{aligned}$$

5. (20 points) Let T be the boundary of the triangle in the plane with vertices $(0,0)$, $(5,0)$, $(3,5)$ (positively oriented). Compute

$$\int_T \cos(x^2)\,dx + xy\,dy$$

Hint: Use Green's theorem.

Solution: By Green's theorem the integral is equal to

$$\iint_R \Big(\frac{\partial xy}{\partial x} - \frac{\partial \cos(x^2)}{\partial y}\Big)\,dx\,dy = \iint_R y\,dx\,dy,$$

where R is the interior of the triangle.

We can compute this integral by explicitly writing out the (cartesian) bounds. For that we need to find the equation of two upper sides of the triangle.

The line that crosses $(0,0)$ and $(3,5)$ has slope $5/3$ and starts at 0, so for that one we have

$$y = \frac{5}{3}x \implies x = \frac{3}{5}y.$$

For the line that crosses $(3,5)$ and $(5,0)$ the slope is $-5/2$, so

$$y = -(5/2)(x-3) + 5 \implies x = 3 + \frac{2(5-y)}{5}$$

With all of this we can now write

$$\begin{aligned}\iint_R y\,dx\,dy &= \int_0^5 \int_{\frac{3}{5}y}^{3-\frac{2(5-y)}{5}} y\,dx\,dy \\ &= \int_0^5 y\Big(3 - \frac{2(5-y)}{5} - \frac{3}{5}y\Big)\,dy \\ &= \int_0^5 y\big(1+y\big)\,dy \\ &= \int_0^5 y + y^2\,dy \\ &= \frac{5^2}{2} + \frac{5^3}{3}.\end{aligned}$$