Math 2374 - Midterm 1

Name: ______________________________

Section: ______

Student ID: ___________________________

Signature: ____________________________

- This exam consists of 6 pages (including this one) and 5 questions.
- Do not give numerical approximations to quantities such as $\sin 5$, π, or $\sqrt{2}$. However, you should simplify $\cos \pi/4 = \sqrt{2}/2$, $e^0 = 1$, and so on.
- Show your work, in a reasonably neat and coherent way, in the space provided. All answers must be justified by valid mathematical reasoning, including the evaluation of definite and indefinite integrals. To receive full credit on a problem, you must show enough work so that your solution can be followed by someone without a calculator.
- Mysterious or unsupported answers will not receive full credit. Your work should be mathematically correct and carefully and legibly written.
- A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations will receive partial credit.
- Full credit will be given only for work that is presented neatly and logically; work scattered all over the page without a clear ordering will receive from little to no credit.

1	2	3	4	5	Total

1. (20 points) Compute the area of the triangle with vertices

$$A = (1, 2), \quad B = (2, 3), \quad C = (3, 5).$$

Solution: Take any of the points, let's say A and compute the vectors to the other two:

$$u = \overrightarrow{AB} \quad \text{and} \quad v = \overrightarrow{AC} :$$

$$u = (1, 1) \quad \text{and} \quad v = (2, 3).$$

Now you can take the absolute value of the determinant to find the area of the parallelogram that the vectors span:

$$\left| \det \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix} \right| = |1 \cdot 3 - 2 \cdot 1| = 1.$$

Since the triangle is half of the spanned parallelogram one just needs to take half of this, so the answer is $\frac{1}{2}$.

2. (20 points) Given the function

$$f(x, y) = \frac{x^2 - y^2}{x^2 + y^2}$$

Can one define $f(0,0)$ so that the f is continuous there? Explain your answer.

Solution: We can approach the point $(0,0)$ along the line L with slope m: $y = mx$. Along this line we have

$$\begin{aligned}
\lim_{\substack{(x,y)\to(0,0)\\ \text{along } L}} f(x, y) &= \lim_{x\to 0} f(x, mx) \\
&= \lim_{x\to 0} \frac{x^2 - (mx)^2}{x^2 + (mx)^2} \\
&= \lim_{x\to 0} \frac{x^2(1 - m^2)}{x^2(1 + m^2)} \\
&= \lim_{x\to 0} \frac{1 - m^2}{1 + m^2} \\
&= \frac{1 - m^2}{1 + m^2}.
\end{aligned}$$

Since this limit depends on the slope it follows that the function cannot be defined at $(0,0)$ to be continuous.

3. (20 points) Consider the function

$$f(x,y) = e^{x-y} + xy.$$

Compute the equation of the plane tangent to the graph of f at the point $P = (3, 2, e+6)$.

Solution: We first compute the gradient of f at the point $(3, 2)$:

$$\begin{aligned} \nabla f(x,y) &= (e^{x-y} + y, -e^{x-y} + x) \\ \implies \nabla f(3,2) &= (e+2, 3-e). \end{aligned}$$

This gives us the coefficients of the linear approximation/tangent plane:

$$L(x,y) = (e+2)(x-3) + (3-e)(y-2) + e + 6,$$

so the tangent plane is

$$z = (e+2)(x-3) + (3-e)(y-2) + e + 6.$$

4. (20 points) Consider the function

$$f(x, y) = x^2 - y^2.$$

- Compute the gradient $\nabla f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})$ at the point $(1, 2)$.

Solution:

$$\begin{aligned} &\nabla f(x, y) = (2x, -2y) \\ \implies &\nabla f(1, 2) = (2, -4). \end{aligned}$$

- Find a unit vector that is orthogonal to $\nabla f(1, 2)$.
Remember that a vector is called a unit vector when its length is one.

Solution: We want to find a unit vector $v = (a, b)$ such that

$$v \cdot (2, -4) = 0$$

or in other words

$$2a - 4b = 0 \iff a = 2b.$$

One can easily find solutions to this equation if we ignore the fact that the vector should be a unit vector, for example $(2, 1)$ would work. The only problem is that it's not normalized, but that can be fixed by dividing by its length:

$$v = \frac{1}{\sqrt{2^2 + 1^2}}(2, 1) = \left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right).$$

5. (20 points) Given the functions

$$f(x,y) = (x^2 - y^2, 2xy)$$

and

$$g(x,y) = (ye^x, xe^y)$$

compute the matrix of partial derivatives of the composition $f \circ g$ at the point $(1,1)$.

Solution: We first compute $g(1,1) = (e,e)$.

Then we compute the matrices of partial derivatives of f and g:

$$Df = \begin{pmatrix} 2x & -2y \\ 2y & 2x \end{pmatrix} \quad \text{and} \quad Dg = \begin{pmatrix} ye^x & e^x \\ e^y & xe^y \end{pmatrix}.$$

Next we evaluate at $g(1,1)$ and $(1,1)$ respectively:

$$Df_{|(e,e)} = \begin{pmatrix} 2e & -2e \\ 2e & 2e \end{pmatrix} \quad \text{and} \quad Dg_{|(1,1)} = \begin{pmatrix} e & e \\ e & e \end{pmatrix}.$$

And finally we multiply them:

$$\begin{pmatrix} 2e & -2e \\ 2e & 2e \end{pmatrix} \cdot \begin{pmatrix} e & e \\ e & e \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 4e^2 & 4e^2 \end{pmatrix}$$