Practice problems for Taylor's theorem

1 What are we talking about?

The linear approximation theorem that we talked about at the beginning of the course can be improved.

Suppose that $f : \mathbb{R}^2 \to \mathbb{R}$ is a smooth scalar function, then we saw that

$$f(x) \approx f(a) + \nabla f(a) \cdot (x - a),$$

this is the linear approximation, and sometimes it is also called the first order (or linear) Taylor approximation.

The second order (or quadratic) Taylor approximation is:

$$f(x) \approx f(a) + \nabla f(a) \cdot (x - a) + \frac{1}{2}(x - a)^T \operatorname{Hess} f(a) \cdot (x - a).$$

There is a lot to unpack here, so let's start with $\operatorname{Hess} f$. This the *Hessian* of f, which is just the matrix of second partial derivatives:

$$\operatorname{Hess} f(a) = \begin{pmatrix} \frac{\partial^2 f(a)}{\partial x \partial x} & \frac{\partial^2 f(a)}{\partial y \partial x} \\ \frac{\partial^2 f(a)}{\partial x \partial y} & \frac{\partial^2 f(a)}{\partial y \partial y} \end{pmatrix}.$$

Remember that here we're taking *second* partial derivatives, which you compute by taking the partial derivatives one after the other. Le't see an example: if $f(x, y) = e^{xy}$ then

$$\begin{aligned} \frac{\partial^2 f(a, b)}{\partial x \partial y} &= \frac{\partial}{\partial x}\Big(\frac{\partial f(a, b)}{\partial y}\Big) \\ &= \frac{\partial}{\partial x}\big(xe^{xy}\big) \\ &= e^{xy} + xye^{xy}. \end{aligned}$$

Remember that for smooth functions the mixed partial derivatives can be taken in any order, i.e.:

$$\frac{\partial^2 f(a)}{\partial y \partial x} = \frac{\partial^2 f(a)}{\partial x \partial y}.$$

Also, sometimes people will write how many derivatives are taken with respect to a variable by putting it in the exponent. For example, two derivatives in x and one in y is sometimes denoted by:

$$\frac{\partial^3 f}{\partial y \partial x^2}.$$

Let's see a full example. Suppose $f(x,y) = \cos(x+y) + xy^2$. To compute the Hessian we first compute the gradient:

$$\nabla f(x,y) = \big(-\sin(x+y) + y^2, -\sin(x+y) + 2xy\big).$$

From this we can compute the Hessian:

$$\begin{aligned}\operatorname{Hess} f(x,y) &= \begin{pmatrix} \frac{\partial}{\partial x}(-\sin(x+y)+y^2) & \frac{\partial}{\partial y}(-\sin(x+y)+y^2) \\ \frac{\partial}{\partial x}(-\sin(x+y)+2xy) & \frac{\partial}{\partial y}(-\sin(x+y)+2xy) \end{pmatrix} \\ &= \begin{pmatrix} -\cos(x+y) & -\cos(x+y)+2y \\ -\cos(x+y)+2y & -\cos(x+y)+2x \end{pmatrix}.\end{aligned}$$

So for example at $(0,\pi/2)$ the Hessian is

$$\operatorname{Hess} f(0,\pi/2) = \begin{pmatrix} 0 & \pi \\ \pi & 0 \end{pmatrix}.$$

So what if we want to compute the quadratic taylor approximation of f at $(0,\pi/2)$? Now we have all the ingredients:

$$f(0,\pi/2) = 0, \quad \nabla f(0,\pi/2) = (-1+\pi^2/4, -1), \quad \operatorname{Hess} f(0,\pi/2) = \begin{pmatrix} 0 & \pi \\ \pi & 0 \end{pmatrix},$$

so

$$f(x,y) \approx (-1+\pi^2/4,-1)\cdot(x,y-\pi/2) + \frac{1}{2}(x,y-\pi/2)^T \begin{pmatrix} 0 & \pi \\ \pi & 0 \end{pmatrix} (x,y-\pi/2).$$

Remember that vectors are represented by columns, so

$$\begin{aligned}(x,y-\pi/2)^T \begin{pmatrix} 0 & \pi \\ \pi & 0 \end{pmatrix} (x,y-\pi/2) &= \begin{pmatrix} x & y-\pi/2 \end{pmatrix} \begin{pmatrix} 0 & \pi \\ \pi & 0 \end{pmatrix} \begin{pmatrix} x \\ y-\pi/2 \end{pmatrix} \\ &= \begin{pmatrix} x & y-\pi/2 \end{pmatrix} \begin{pmatrix} \pi y - \pi^2/2 \\ \pi x \end{pmatrix} \\ &= \pi xy - \pi^2 x/2 + \pi xy - \pi^2 x/2 \\ &= 2\pi xy - \pi^2 x.\end{aligned}$$

On the other hand

$$(-1+\pi^2/4,-1)\cdot(x,y-\pi/2) = -x + \pi^2 x/4 - y + \pi/2,$$

so putting it all together:

$$f(x,y) \approx -x + \pi^2 x/4 - y + \pi/2 + \pi xy - \pi^2 x/2.$$

Here are some more resources:

1. Idea: `https://mathinsight.org/taylors_theorem_multivariable_introduction`
2. More examples: `https://mathinsight.org/taylor_polynomial_multivariable_examples`
3. This is one-dimensional, but it's still good to watch: `https://www.youtube.com/watch?v=3d6DsjIBzJ4`

2 Quick cheat sheet

$$f(x) \approx f(a) + \nabla f(a) \cdot (x-a) + \frac{1}{2}(x-a)^T \operatorname{Hess} f(a) \cdot (x-a).$$

$$\operatorname{Hess} f(a) = \begin{pmatrix} \frac{\partial^2 f(a)}{\partial x \partial x} & \frac{\partial^2 f(a)}{\partial y \partial x} \\ \frac{\partial^2 f(a)}{\partial x \partial y} & \frac{\partial^2 f(a)}{\partial y \partial y} \end{pmatrix}.$$

3 Some example questions

1. (2009 Spring final) Consider the function

$$f(x,y) = e^{2x+2y+2}(5y - y^2 - 3)$$

   Give the linear and quadratic approximations of f near the point $(-2, 1)$.

2. (2007 Spring final) Consider the surface defined by $z = f(x,y) = 2x^2 + xy + y^2 - 3$. Find the quadratic approximation of the surface (i.e.: the second-order Taylor polynomial of f) at the point $(x,y,z) = (1,2,5)$. Use your approximation to estimate the value of $f(0.8, 2.1)$.