Practice problems for Stokes theorem

1 What are we talking about?

Stokes theorem says

$$\iint_S \operatorname{curl} V \cdot dS = \int_{\partial S} V \cdot ds,$$

where S is an oriented surface and ∂S is the boundary of S with the orientation induced by S.

What does this last phrase mean? An orientation of S is a consistent (continuous) way of assigning unit normal vectors $\vec{n}$. The orientation of ∂S is the one in which the surface would be to your left if you were walking along the boundary on the positive side of S (so, if yor feet are touching S and you draw a vector from your feet to your head then this vector should point in the same direction as $\vec{n}$).

For example, consider the northern hemisphere

$$x^2 + y^2 + z^2 = 1 \quad \text{and} \quad z \geq 0,$$

with the orientation given by the *outer* pointing unit normal, i.e.: $n(x, y, z) = (x, y, z)$. The boundary is the circle of radius 1 on the XY plane:

$$x^2 + y^2 = 1 \quad \text{and} \quad z = 0.$$

To see what the orientation should be imagine you're walking along the boundary (this circle above). Since the orientation of the surface is the outer pointint unit normal, in order for the surface to stay on your left you should walk counterclockwise.

So a parametrization of the boundary with the correct orientation is

$$\gamma(t) = (\cos t, \sin t, 0).$$

If instead we were looking at the *southern* hemisphere the boundary would be the same but the orientation would be the opposite (draw a picture!).

Here are some more resources:

1. Idea: `https://mathinsight.org/stokes_theorem_idea`
2. Orientation: `https://mathinsight.org/stokes_theorem_orientation`
3. More examples: `https://mathinsight.org/stokes_theorem_examples`
4. MIT OCW lecture about Stoke's theorem: `https://www.youtube.com/watch?v=tzoYhe3H5dM`

2 Quick cheat sheet

$$\iint_S \operatorname{curl} V \cdot dS = \int_{\partial S} V \cdot ds,$$

where the boundary ∂S of S is oriented so that the surface is to your left when walking on the positive side of S.

Another way of writing this is:

$$\iint_S \operatorname{curl} V \cdot \vec{n}\, dS = \int_{\partial S} V \cdot ds,$$

You can use this to go from integrals over surfaces to integrals over curves and back.

3 Some example questions

1. (2011 midterm 3) Let C be the oriented curve (the boundary of a triangle) which moves in straight lines from $(0,0,0)$ to $(2,0,0)$ to $(0,0,1)$ and back to $(0,0,0)$, in that order. Use Stokes' Theorem to calculate the line integral $\int_C V \cdot ds$, where

$$V(x,y,z) = \left(-y^2 z, e^{xz}, xy - \sqrt{z^2+1}\right).$$

Solution: If we want to use Stokes' theorem we need a surface (not just the curve C). So we can "fill in" the triangle and get a surface T which is the portion of the plane induced by those points that lies inside the triangle.

Then Stokes' theorem says:

$$\int_C V \cdot ds = \iint_T \operatorname{curl} V \cdot dS,$$

provided we orient T so that the induced orientation on the boundary coincides with the one given in the problem.

Let's first find a parametrization of T. If we give names to the vertices of the triangle:

$$A = (0,0,0) \quad B = (2,0,0) \quad C = (0,0,1),$$

then we can find two vectors in the plane, whose cross product will give us the normal vector to the plane. In particular

$$u = AB = (2,0,0) \quad v = AC = (0,0,1) \implies u \times v = (0,-2,0).$$

Since the plane goes through the origin, we can use the normal vector for of the plane equation to get the equation of the plane in which T sits:

$$(x,y,z)\cdot(0,-2,0) = 0 \implies -2y = 0 \implies y = 0.$$

So the plane is just the XZ plane (which we could also have guessed since A, B, and C are all in the XZ plane!).

Having the equation of the plane makes the parametrization easy:

$$\Phi(u,v) = (u,0,v),$$

where u and v lie in the domain D which is the triangle (in the plane) with vertices

$$(0,0) \quad (2,0) \quad (0,1).$$

With this parametrization we have an orientation of the plane:

$$T_u = (1,0,0) \quad T_v = (0,0,1) \implies n = \frac{T_u \times T_v}{\|T_u \times T_v\|} = (0,-1,0).$$

Is this orientation the one that induces the orientation of C given in the problem? This vector is pointing "out" of the first octant in a direction perpendicular to the XZ plane (which is where C lives). So, if we walk along C on the positive side of XZ (given by this orientation) we will go through the points in the order A, B, C. This is the correct order!!!

Putting (almost) everything together we arrive at

$$\int_C V \cdot ds = \iint_S \operatorname{curl} V \cdot dS = \iint_D (\operatorname{curl} V) \circ \Phi \cdot T_u \times T_v \, du \, dv.$$

So we're left with computing the curl of V:

$$\operatorname{curl} V = \det \begin{pmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ -y^2 z & e^{xz} & xy - \sqrt{z^2+1} \end{pmatrix} = (x - xe^{xz}, -y - y^2, ze^{xz} + 2yz).$$

We can now compute

$$\operatorname{curl} V) \circ \Phi(u,v) = (u - ue^{uv}, 0, ve^0).$$

Thus

$$(\operatorname{curl} V) \circ \Phi \cdot T_u \times T_v = (u - ue^{uv}, 0, z) \cdot (0,-1,0) = 0,$$

and so

$$\int_C V \cdot ds = \iint_S \operatorname{curl} V \cdot dS = \iint_D (\operatorname{curl} V) \circ \Phi \cdot T_u \times T_v \, du \, dv = 0.$$

2. (2007 Spring final exam) Let S be the paraboloid $z = (x^2+y^2)/4$ for $z \leq 4$ oriented with upward normal vector. Use Stokes' Theorem to calculate $\iint_S \operatorname{curl} V \cdot dS$, where

$$V(x,y,z) = xy^2 z \vec{i} - 4x^2 y \vec{j} + \frac{z-1}{x^2+2y^2+1} \vec{k}.$$

Solution: The orientation induced by the upward pointing normal gives the *counterclockwise* orientation to the boundary of S (the circle of radius 4 centered at $(0, 0, 4)$ parallel to the XY plane).

So, by Stokes' theorem:

$$\iint_S \operatorname{curl} V \cdot dS = \int_C V \cdot ds,$$

where C is the boundary circle oriented as mentioned above. Since we're giving C the counterclockwise orientation we parametrize it by

$$\gamma(t) = (4\cos t, 4\sin t, 4) \quad t \in [0, 2\pi].$$

Now V evaluated at $\gamma(t)$ looks like:

$$\begin{aligned} V(\gamma(t)) &= \left(4\cos t(4\sin t)^2 4, -4(4\cos t)^2 4\sin t, \frac{3}{17 + (4\sin t)^2}\right) \\ &= \left(4^4\cos t\sin^2 t, -4^4\cos^2 t\sin t, \frac{3}{17 + 4^2\sin^2 t}\right). \end{aligned}$$

We need to take the dot product of this with γ':

$$\gamma'(t) = (-4\sin t, 4\cos t, 0),$$

so the last component of V doesn't matter (phew!), and we get

$$\begin{aligned} V(\gamma(t)) \cdot \gamma'(t) &= 4^4\cos t\sin^2 t \cdot (-4\sin t) + -4^4\cos^2 t\sin t \cdot 4\cos t \\ &= -4^5\cos t\sin^3 t - 4^5\cos^3 t\sin t \\ &= -4^5\cos t\sin t(\sin^2 t + \cos^2 t) \\ &= -4^5\cos t\sin t \\ &= -2^9\sin(2t). \end{aligned}$$

Finally, putting it all together:

$$\begin{aligned} \iint_S \operatorname{curl} V \cdot dS &= \int_C V \cdot ds \\ &= \int_0^{2\pi} -2^9\sin(2t)\, dt \\ &= 0. \end{aligned}$$

3. (Spring 2006 final) Let C_1 be the circle of radius 1 in the plane $z = 1$ and C_2 be the circle of raiuds 1 in the plane $z = 2$. These curves are parametrized by $\vec{c_1}(t) = (\cos t, \sin t, 1)$ and

$\vec{c_2}(t) = (\cos t, \sin t, 2)$, for $0 \leq t \leq 2\pi$. If V is a vector field whose curl is

$$\nabla \times V = (1 - x^2 - y^2)(x\vec{i} + y\vec{j}) + (y\vec{i} - x\vec{j}) + xe^{3z}\vec{k},$$

show that the line integrals are equal:

$$\int_{C_1} V \cdot ds = \int_{C_2} V \cdot ds.$$

4. If C is the triangle with vertices $(2, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 3)$ oriented counterclockwise as viewed from above. Use Stokes' theorem to compute

$$\int_C (5x^2, x, z^3) \cdot ds.$$

5. Let S be the southern hemisphere of $x^2 + y^2 + z^2 = 1$ oriented with the outward pointing unit normal. Let V be the vector field given by

$$(x^2 + y^2 - z^2)(x\vec{i} + y\vec{j} + \vec{k}).$$

Use Stoke's theorem to compute

$$\iint_S \operatorname{curl} V \cdot dS.$$