Practice problems for Gauss' theorem

1 What are we talking about?

Gauss' theorem (aka the Divergence theorem) says

$$\iiint_W \operatorname{div} F \, dV = \iint_{\partial W} V \cdot dS,$$

where W is a solid simple region (think a ball, for example), and ∂W is the *boundary* of W with the *outward pointing normal*.

This "induced orientation" is easier than what we had to do in Stokes' theorem: in this case we don't need to look left or right, or use the right hand rule or anything. The orientation on ∂W is *always* the one that points out.

This is also a generalization of the fundamental theorem of calculus: we're trading a bunch of derivatives ($\operatorname{div} V$) to get one fewer integral but over the boundary.

Here are some more resources:

1. Idea: https://mathinsight.org/divergence_theorem_idea
2. More examples: https://mathinsight.org/divergence_theorem_examples
3. MIT OCW lecture about the divergence theorem (he talks about this towards the end, but there are some good exercises before): https://www.youtube.com/watch?v=WfEQabCGAqI
4. MIT OCW rectiation about the divergence theorem: https://www.youtube.com/watch?v=CCoTAyZ14XM

2 Quick cheat sheet

$$\iiint_W \operatorname{div} F \, dV = \iint_{\partial W} V \cdot dS = \iint_{\partial W} V \cdot \vec{n} \, dS,$$

where W is a solid simple region (think a ball, for example), and ∂W is the *boundary* of W with the *outward pointing normal*.

You can use this to go from surface integrals to solid integrals and back.

3 Some example questions

1. (2006 Spring final) Let W be the rectangular solid $[0,1] \times [0,4] \times [0,1/2]$ in $\mathbb{R}^3$. Write S for the boundary surface of W oriented with the outward unit normal vector. Let F be the vector field

$$F = (x^2 + ze^{y^2})\vec{i} + (x\sin(\pi z) - xy)\vec{j} + (3z - xz + x^4 \log y)\vec{k},$$

where log is the natural logarithm.

Use the divergence theorem to compute the flux integral

$$\iint_S F \cdot dS.$$

> **Solution:** Gauss' theorem tells us that we can just compute the triple integral of the divergence of F over W. So let's compute the divergence first:
>
> $$\begin{aligned} \operatorname{div} F &= \frac{\partial}{\partial x}(x^2 + ze^{y^2}) + \frac{\partial}{\partial y}(x\sin(\pi z) - xy) + \frac{\partial}{\partial z}(3z - xz + x^4 \log y) \\ &= 2x + -x + 3 - x \\ &= 3. \end{aligned}$$
>
> (thank you test writer!!!)
>
> So, applying Gauss' theorem:
>
> $$\iint_S F \cdot dS = \iiint_W 3\, dV = 3\,\text{Volume}(W) = 6.$$

2. Let F be the vector field given by

$$F = (1 - x^2 - y^2 - z^2)(e^{x^2}\vec{i} + e^{y^2}\vec{j} + e^{z^2}\vec{k}).$$

Use Gauss' theorem to compute

$$\iiint_B \operatorname{div} F\, dV,$$

where B is the unit ball $x^2 + y^2 + z^2 \leq 1$.

> **Solution:** We're not even going to compute the divergence of F because it's going to be a mess, and because thanks to Gauss' theorem we don't have to. Instad, by Gauss' theorem:
>
> $$\iiint_B \operatorname{div} F\, dV = \iint_S F \cdot n\, dS,$$
>
> where S is the boundary of B, i.e.: the sphere $x^2 + y^2 + z^2 = 1$.

Now observe that F is identically 0 on S, since $1 - x^2 - y^2 - z^2 = 0$, so

$$\iint_S F \cdot n \, dS = 0$$

(we didn't even need to find a parametrization of S, or compute the normal vector).

So the final answer is 0.

3. (Fall 2010 final) Let F be the vector field

$$F = (e^y \sin x - 3xz^2, \cos y - e^y \cos x, z^3 + z \sin z).$$

Compute $\iint_S F \cdot dS$, where S is the part of the cylinder $y^2 + z^2 = 4$ between the planes $x = 0$ and $x = 9$.

4. Evaluate

$$\iint_S F \cdot dS,$$

where S is the unit sphere and

$$F = (3xy^2, 3x^2y, z^3).$$

5. Suppose F is tangent to the closed surface $S = \partial W$, where W is a closed region in $\mathbb{R}^3$. Prove that

$$\iiint_W \operatorname{div} F \, dV = 0.$$