Practice problems for local extrema

1 What are we talking about?

We want to maximize functions, just like in single variable calculus, but now in dimension two. Let's recall first what happened in one dimension.

If $f : [a, b] \to \mathbb{R}$ is a smooth function and $c \in (a, b)$ is a *local extrema* (a local maximum or minimum) then $f'(c) = 0$. We called the points where $f'(c) = 0$ *critical points*. In order to see whether a critical point is a local maximum or a local minimum we took the second derivative and checked its sign: positive meant local minimum, negative meant local maximum, and zero meant we needed higher order derivatives.

The situation in two dimensions is more complicated. Let say that you have a function $f : D \to \mathbb{R}$ defined on some (closed) domain D. Just like in one dimension local extrema happen on *critical points*:

$$(x_0, y_0) \text{ is a critical point if } \nabla f(x_0, y_0) = 0.$$

How do we decide if this is a local maximum or a local minimum? We can use Taylor's theorem: if (x_0, y_0) is a critical point then for (x, y) close to (x_0, y_0)

$$f(x, y) \approx f(x_0, y_0) + \frac{1}{2}(x - x_0, y - y_0)^T \operatorname{Hess} f(x_0, y_0)(x - x_0, y - y_0)$$

(there is no linear term because $\nabla f(x_0, y_0) = 0$).

If we could somehow show that

$$\frac{1}{2}(x - x_0, y - y_0)^T \operatorname{Hess} f(x_0, y_0)(x - x_0, y - y_0) > 0$$

for all $(x, y) \neq (x_0, y_0)$ then the point (x_0, y_0) should be a local minimum (since all nearby points are higher).

Similarly, if

$$\frac{1}{2}(x - x_0, y - y_0)^T \operatorname{Hess} f(x_0, y_0)(x - x_0, y - y_0) < 0$$

for all $(x, y) \neq (x_0, y_0)$ then the point (x_0, y_0) should be a local maximum (since all nearby points are lower).

So we need to investigate the properties of $\operatorname{Hess} f(x_0, y_0)$. Here are the most comon cases, and for simplicity we will just assume the matrix has the form

$$\operatorname{Hess} f(x_0, y_0) = H = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

(Indefinite) $\det(H) = ac - b^2 < 0$. Then the point is a *saddle point*. In this case some nearby points will be larger and some smaller.

(Positive definite) $\det(H) > 0$ and $a > 0$. Then the point is a local minimum.

(Negative definite) $\det(H) > 0$ and $a < 0$. Then the point is a local maximum.

Here are some more resources:

1. Idea: https://mathinsight.org/local_extrema_introduction_two_variables
2. More examples: https://mathinsight.org/local_extrema_examples_two_variables

2 Quick cheat sheet

A point (x_0, y_0) is a critical point if $\nabla f(x_0, y_0)$. Local extrema happen at critical points but not all critical points are necessarily local extrema.

One can use the Hessian to check (in some cases) if a critical point is a local maximum or a local minimum (or a saddle point). If

$$\text{Hess}\, f(x_0, y_0) = H = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

then the most common cases are

(Indefinite) $\det(H) = ac - b^2 < 0$. Then the point is a *saddle point*. In this case some nearby points will be larger and some smaller.

(Positive definite) $\det(H) > 0$ and $a > 0$. Then the point is a local minimum.

(Negative definite) $\det(H) > 0$ and $a < 0$. Then the point is a local maximum.

3 Some example questions

1. Classify the critical points of the function

$$f(x, y) = x(x^2 + y^2 - 1)$$

Solution: We first need to find what the critical points are:

$$\nabla f(x, y) = (3x^2 + y^2 - 1, 2xy).$$

For this to be 0 the last coordinate has to be 0, so $x = 0$ and/or $y = 0$. Also, if both are 0 then the first coordinate is $-1 \neq 0$, so there are only two possibilities:

- If $x = 0$ and $y \neq 0$.

  Then the gradient ends up being $(y^2 - 1, 0)$, which has solutions $y = \pm 1$, so from here we get two critical points:

  $$A = (0, -1) \quad B = (0, 1).$$

- If $x \neq 0$ and $y = 0$.

  Then the gradient is $(3x^2 - 1, 0)$, so we have $x = \pm 1/\sqrt{3}$. So we get another pair of critical points

  $$C = \left(-\frac{1}{\sqrt{3}}, 0\right) \quad D = \left(\frac{1}{\sqrt{3}}, 0\right).$$

So we have these four critical points, now we need to check the Hessian at these points. The Hessian at a generic point is

$$\operatorname{Hess} f(x, y) = H = \begin{pmatrix} 6x & 2y \\ 2y & 6x \end{pmatrix},$$

so now we can just evaluate this at each point:

- At A

  $$\operatorname{Hess} f(A) = \begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix}.$$

  The determinant is negative, so this is a saddle point.

- At B

  $$\operatorname{Hess} f(B) = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}.$$

  The determinant is negative, so this is a saddle point.

- At C

  $$\operatorname{Hess} f(C) = \begin{pmatrix} -\frac{6}{\sqrt{3}} & 0 \\ 0 & -\frac{2}{\sqrt{3}} \end{pmatrix}.$$

  The determinant is positive and the first entry is negative, so this is a local maximum.

- At D

  $$\operatorname{Hess} f(D) = \begin{pmatrix} \frac{6}{\sqrt{3}} & 0 \\ 0 & \frac{2}{\sqrt{3}} \end{pmatrix}.$$

  The determinant is positive and the first entry is positive, so this is a local minimum.

2. Classify the critical points of the function

$$f(x, y) = x^3 + x^2 y - y^2 - 4y.$$

3. Classify the critical points of the function

$$f(x, y) = xy + \frac{16}{y} + \frac{32}{x}.$$

4. Classify the critical points of the function

$$f(x, y) = x^3 - y^3 + 8xy.$$

5. Classify the critical points of the function

$$f(x, y) = xye^{-8(x^2+y^2)}.$$